MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG and Research Department of Mathematics

II B.Sc. Mathematics– Semester - III

E-Notes (Study Material)

Elective: Mathematical Statistics I	**Code:23UEMA33**
Unit:4: Poisson distribution: Properties, Moments of Poisson distribution– Geometric distribution: Moment generating function of Geometric distribution.	
Learning Objectives: To relate the statistical distributions with the real life situations.	
Course Outcome: illustrate the theory of random variables, distribution functions and probability distributions with suitable.	

Overview: Poisson distribution, Geometric Distribution, Moment generating function

Unit IV

Poisson Distribution.

Condition.

A Random variable x is said to be Poisson Distribution If it assumes only Non negative values and its probability mass function is given by

$$P(x, \lambda) = P(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^{x}}{x!} & ; x = 0, 1, 2, \ldots, \lambda = 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Where,

λ is known as parameter of the distribution

Example:

$x \sim p(\lambda)$ parameter denotes that x is a poisson variate with parameter λ

Note:

1. $$\sum_{x=0}^{\infty} p(X=x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1 \qquad \because \sum \frac{\lambda^{x}}{x!} = e^{\lambda}$$

Conditions for poisson Distribution.

1. No of death, such as heart attach by due to snake bite, etc...

2. No of suicide in a particular city.

3. No. of printing mistake in each page of the book.

4. No. of car crossing for a minute during the busy of hours of a day.

Moments of poisson Distributions.

First four moments about origin of poisson Distributions are attained as follows.

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{n}}{x!}$$

$$= \sum_{x=0}^{\infty} x \cdots \frac{e^{-\lambda}\lambda^{x}}{x(x-1)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}\lambda^{-1}}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[\frac{\lambda^{-1}}{(-1)!} + \frac{\lambda^{0}}{0!} + \frac{\lambda^{1}}{1!} + \cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda} \qquad e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} +$$

$$\mu_1' = \lambda$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=0}^{\infty} (x^2 + x - x)\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda}\lambda^{x}}{x(x-1)!} + \lambda$$

$$= \sum_{x=0}^{\infty} (x-1) \frac{e^{-\lambda}\lambda^x}{(x-1)(x-2)!} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} + \lambda = e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{\lambda^x}{(x-2)!}\right] + \lambda$$

$$= e^{-\lambda}\left[\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \dots\right] + \lambda = e^{-\lambda}\lambda^2\left[e^{\lambda}\right] + \lambda$$

$$\mu_2' = \lambda^2 + \lambda$$

$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

Central moments :-

$$\mu_1 = \mu_1' - \mu_1' = \lambda - \lambda = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3(\lambda)(\lambda^2 + \lambda) + 2\lambda^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda^3 - 3\lambda^2 + 2\lambda^3$$

$$\mu_3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 - 4(\lambda^3 + 3\lambda^2 + \lambda)\lambda +$$
$$6(\lambda^2 + \lambda)\lambda^2 - 3\lambda^4$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda + \lambda - 4\lambda^4 - 12\lambda^3$$
$$- 4\lambda^2 + 6\lambda^4 + 6\lambda^3 - 3\lambda^4$$

$$\mu_4 = 3\lambda^2 + \lambda$$

Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2+\lambda}{\lambda^2} = \frac{3\lambda^2}{\lambda^2} + \frac{\lambda}{\lambda^2}$$

$$= 3 + 1/\lambda$$

Moment generating function of Poisson distribution.

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$M_x(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{e^t\lambda}{1!} + \frac{e^{2t}\lambda^2}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[e^{e^t \lambda}\right] = e^{-\lambda}\left[e^t\lambda\right]$$

$$M_x(t) = e^{\lambda}\left[e^t - 1\right] \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Additive Properties of poisson distribution (or) Reproductive property of independent of poisson variate.

Statement :

Sum of independent poisson variate is also a poisson variate i.e, if $x_i (i = 1, 2, \ldots n)$ are independent Poisson variate with parameter λ_i $(i = 1, 2, \ldots, n)$ respectively. Then $\sum_{i=1}^{n} x_i$ is also a poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$

Proof

$$M_{\sum_{i=1}^{n} x_i}(t) = M_{(x_1 + x_2 + \dots + x_n)}(t)$$

$$= M_{x_1}(t) . M_{x_2}(t) \dots M_{x_n}(t)$$

$$= e^{\lambda_1(e^t-1)} . e^{\lambda_2(e^t-1)} \dots e^{\lambda_n(e^t-1)}$$

$$= e^{(e^t-1)\left(\sum_{i=1}^{n} \lambda_i\right)} \qquad M_x(t) = e^{\lambda(e^t-1)}$$

Which is the moment generating function of poisson variate with parameter $\lambda_1, \lambda_2, \dots \lambda_n$

Hence by uniqueness theorem of moment generating function, $\sum_{i=1}^{n} x_i$ is also a poisson variate.

Parameter $\sum_{i=1}^{n} \lambda_i$

Remark

The difference of two independent Poisson variate is not a poisson variate

$$M_{x_1 - x_2}(t) = M_{x_1}(t) . M_{x_2}(t)$$

$$= (M_{x_1} . M_{x_2}) t = M_{x_1}(t) . M_{x_2}(-t)$$

$$= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^{-t}-1)}$$

$$= e^{\lambda_1(e^t-1) + \lambda_2(e^{-t}-1)}$$

which cannot be putting the form $e^{\lambda(e^t-1)}$

$\therefore x_1 - x_2$ is not a poisson variate.

Recurrance Formula for the Probability of poisson distribution (or) fitting of poisson distribution.

The poisson distributing with λ is defind by

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$p(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \rightarrow (2)$$

$$(2) \div (1) \Rightarrow \frac{P(x+1)}{P(x)} = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \times \frac{x!}{e^{-\lambda}\lambda^{x}}$$

$$= \frac{e^{-\lambda}\lambda^{x}.\lambda}{(x+1)!} \times \frac{x!}{e^{-\lambda}\lambda^{x}}$$

$x! = x(x+1)!$

$(x+1)! = (x+1)$

$(x+1-1)!$

$$\frac{P(x+1)}{P(x)} = \frac{\lambda}{(x+1)!} \times x! = \frac{\lambda\, x!}{(x+1)x!}$$

$$P(x+1) = \frac{\lambda}{(x+1)} P(x)$$

Which is the required recurrance formula only probability we need the Calculate $P(0)$ is given by

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-\lambda}$$

Where λ is estimated using given data

i.e. mean $= \lambda$

The another Probability is $P(1), P(2) \ldots$ be easily obtain as follows

$$P(x+1)_{x=0} \Rightarrow P(1) = \frac{\lambda}{1} P(0)$$

$$P(1) = \lambda e^{-\lambda}$$

$P(x+1)_{x=1} \Rightarrow P(2) = \frac{\lambda}{2} P(1)$

$$P(2) = \frac{\lambda}{2} \lambda e^{-\lambda}$$

$P(x+1)_{x=2} \Rightarrow P(3) = \frac{\lambda}{3} \frac{\lambda^2 e^{-\lambda}}{2}$

$$P(3) = \frac{\lambda^3 e^{-\lambda}}{6}$$

Mode of the poisson distribution.

The probability mass function of the poisson distribution is given by

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \rightarrow (1)$$

$$P(x-1) = \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} \rightarrow (2)$$

$(1) \div (2) \Rightarrow \frac{P(x)}{P(x-1)} = \frac{e^{-\lambda} \lambda^x}{x!} \times \frac{(x-1)!}{e^{-\lambda} \lambda^x \lambda^{-1}}$

$$= \frac{(x-1)! \lambda}{x(x-1)!}$$

$$\frac{P(x)}{P(x-1)} = \frac{\lambda}{x}$$

Case (i)

When λ is not an integer let us Suppose that 'S' is the integral part of the λ

$$\frac{P(x)}{P(x-1)} > 1 \text{ for } x = 1, 2, \ldots S$$

$$\frac{P(x)}{P(x-1)} < 1 \text{ for } x = S+1, S+2, \ldots S+n$$

$P(1) > P(0), P(2) > P(1) \ldots P(S) > P(S-1)$

$p(s+1) < p(s)$, $p(s+2) < p(s+1) \dots \infty$

$\Rightarrow p(0) < p(1) < p(2) \dots p(s-1) < p(s)$

$\Rightarrow p(s) > p(s+1) > p(s+2) \dots$

combaining above expression we get

$p(0) < p(1) < p(2) \dots < p(s) > p(s+1) > p(s+2)$

In this case the distribution is unique model and the integral part of λ is unique modal value.

<u>case (ii)</u>

λ is an integer, $\lambda = K$

$$\frac{P(x)}{P(x-1)} = \begin{cases} >1 & \text{for } x = 1/K \text{ (or) } 0, 1, 2, \dots \\ =1 & \text{for } x = K \\ <1 & \text{for } x = K+1, K+2 \end{cases}$$

$\Rightarrow p(1) > p(0)$, $P(2) > P(1)$, $\dots P(K-1) > P(K-2)$

$P(0) < P(1)$, $P(1) < P(2) \dots P(K-2) < P(K-1)$

when $x = K+1, K+2, \dots$

$$\frac{P(x)}{P(x-1)} = 1 \qquad \frac{P(x)}{P(x-1)} < 1$$

When $x = K$

$$\frac{P(K)}{P(K-1)} = 1 \qquad \frac{P(K+1)}{P(K+1-1)} < 1$$

$$\frac{P(K+1)}{P(K)} < 1$$

$$\frac{P(K+2)}{P(K+1)} < 1 \dots \infty$$

$P(K+1) < p(K)$, $P(K+2) < p(K+1) \dots \infty$

combaining the above expression, we get

$P(0) < p(1) < p(2) \dots < P(K-1) = P(K) > P(K+1) > P(K+2)$

Therefore we have two maximum value $P(K-1)$ and $p(K)$ the distribution is bimodal are $K-1$ & K

i.e, which is $\lambda-1$ & λ

$K=\lambda$

<u>Problems:</u>

1) A car hire firm has two cars which its hires out day by day. Then the number of demand on each day is distributed has poisson variate with mean 1.5, calculate the preposition of days on which

i) Neither car is used

ii) Some demand is refused.

Here the random variable x which denotes the no. of demands for a car on any day follows poisson distribution.

with mean $\lambda = 1.5$

The proportion of days which there is x demands for a car is given by $\lambda=1.5$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{1.5}(1.5)^x}{x!} \quad x=0,1,2,\ldots$$

i) Neither car is used

Proportion of days on which neither car is used.

$$P(X=0) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5}$$

$$= 0.2231$$

ii) Some demand is refused

Proportion of days on which Some demand is refused is

$P(x>2) = 1 - P(x \leq 2)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - e^{1.5}\left[1 + 1.5 + \frac{(1.5)^2}{2}\right]$

$P(x>2) = 0.1912$

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2) A manufacture of cotton pins knows that 5% of is product it is defective sales, cotton pins in boxes are 100 and guarentee quality not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the gurentee quality.?

Given:

$n = 100$

Let, P = Probability of defective items 5%

$P = \frac{5}{100} = 0.05$

λ = mean = No. of defective pins in a box of 100

$\lambda = np$

$\lambda = 100 \times 0.05 = 5$

$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-5}(5)^{x}}{x!}, \quad x = 0, 1, 2, \ldots$

Probability of guass will fail to meet the guarentee quality is

$= 100 - 10 \Rightarrow 90 > 10$

$$P(x>10) = 1 - P(x \le 10)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)]$$

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3) An insurrance company insures 4000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00,000 will have an accident each year that result in this type of injury what is the probability that more than 3 of the insured with collect on their policy in a given year?

In usual Notations, given,

$n = 4000$ and $P =$ probability of loss of both eyes in a car accident

$$= \frac{10}{100000} = 0.0001$$

Since p is very small and ... we may approximate the given distribution by poisson distribution thus the parameter λ of the poisson distribution is,

$$\lambda = np = 4000 \times 0.0001 = 0.4$$

Let the random variable x denotes number of car accidents in batch of 4,000 people Then by poisson probability law,

$$p(x = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.4}(0.4)^{x}}{x!} ; x = 0, 1, 2..$$

Hence, The required probability that more than 3 of the insured will collect on their policy is given by,

$$p(x>3) = 1 - [p(x=0) + p(x=1) + p(x=2) + p(x=3)]$$

$$= 1 - e^{-0.4}\left\{(0.4)^{0} + (0.4) + \frac{(0.4)^{2}}{2!} + \frac{(0.4)^{3}}{3!}\right\}$$

$$= 1 - 0.6703(1 + 0.4 + 0.08 + 0.0107)$$

$$= 1 - 0.6703 \times 1.4907$$

$$= 0.0008$$

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2) A Manufacturer, who produces medicine bottles, find that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles using poisson destribution. Find how many boxes will contain.

i) No defective and

ii) at least two defective.

In the usual notations, given : $N=100$, $n=500$, p = probability of a defective bottle $= 0.001$,

$$\lambda = np = 500 \times 0.001 = 0.5$$

Let the random variable X denote the number of defective bottles in a box of 500. Then by poisson probability law, the probability of x defective bottles in a box is given by,

$$P(X=x) = \frac{e^{-0.5}(0.5)^x}{x!} = \frac{0.6065 \times (0.5)^x}{x!} ; \quad x=0,1,2,\ldots$$

Hence in a consignment of 100 boxes, the frequency (number) of boxes containing x defective bottles is,

Two values

$$f(x) = NP(X=x) = \frac{100 \times 0.6065 \times (0.5)^x}{x!}$$

i) Number of boxes containing no defective bottles,

$$= 100 \times P(X=0) = 100 \times 0.6065$$
$$= 61$$

ii) No. of Boxes containing at least two defective bottles, $1-[P(X=0)+P(X=1)]$

so 0,1

$$= 100[P(X \geq 2)] = 100\{1-P(X=0)-P(X=1)\}$$

$= 100(1 - 0.6065 - 0.6065 \times 0.5)$

$= 100 \times 0.09025 = 9$

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3) Six coins are tossed 6400 times using the poisson distribution, find the approximate probability of getting 6 heads 'r' times.

Soln)

The probability of obtaining 6 heads in one throw of 6 coins (a single trial) is, $P = (1/2)^6$, assuming that head and tail are equally probable,

$$\lambda = np = 6400 \times \left(\tfrac{1}{2}\right)^6 = 100$$

Hence, using poisson probability law, the required probability of getting 6 heads r times is given by

$$P(X=r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$= \frac{e^{-100} (100)^r}{r!}, \quad r = 0, 1, 2 \ldots$$

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4) In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

The average no. of typo-graphical errors per page in the book is given by

$$\lambda = \frac{390}{520} = 0.75$$

Hence, By Poisson probability law, the probability of x errors per page is given by

$$P(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2,\ldots$$

The required probability that a random sample of 5 pages will contain no error is

$$[P(x=0)]^5 = \left[\frac{e^{-0.75}(0.75)^0}{0!}\right] = (e^{-0.75})^5$$

$$= e^{-3.75}$$

$$[P(x=0)]^5 = 0.0235$$

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5) In a poisson frequency distribution, Frequency corresponding to three Success is 2/3 times frequency corresponding 4 Success. Find the mean and Standard deviation of the distribution.

Let, x be a random variable and the parameter is λ

The frequency function is given by

$$f(x) = NP(x) = NP[X=x]$$

$$= N.\frac{e^{-\lambda}\lambda^{x}}{x!} ; x = 0, 1, 2, \ldots \rightarrow ①$$

put $x = 3$ in ①

$$f(3) = N.\frac{e^{-\lambda}\lambda^{3}}{3!}$$

put $x = 4$ in ①

$$f(4) = N.\frac{e^{-\lambda}\lambda^{4}}{4!}$$

Given,

$$f(3) = \frac{2}{3}f(4)$$

$$N.\frac{e^{-\lambda}\lambda^{3}}{3!} = N.\frac{2}{3}.\frac{e^{-\lambda}\lambda^{4}}{4!}$$

$$\frac{1}{3!} = \frac{2}{3}\frac{\lambda}{4!}$$

$$\frac{1}{6}\frac{3}{2} = \frac{\lambda}{24}$$

$$\lambda = 6$$

Mean of the poisson distribution is equal to λ

$$\lambda = 6$$

Standard deviation in poisson distribution is $\sqrt{\lambda}$

$$S.D = \sqrt{6}$$

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6) Suppose That the number of telephone Calls coming into a telephone exchange between 10 a.m and 11. am Say, X_1 is a random variable with poisson distribution with parameter two. Similarly The number of calls arriving between 11 am and 12 noon Say, X_2 has a poisson distribution with parameter 6. If X_1 and X_2 are independent, what is the probability that more than 5 calls come in between 10 am and 12 noon.

Given:

$X_1 \sim P(2)$ and $X_2 \sim P(6)$

Let $X = X_1 + X_2$

By the additive property of poisson distribution, X is also a poisson variate with parameter λ.

i.e) $\lambda = 2+6$

$\lambda = 8$

Hence, The probability of x calls in between 10 am and 12 noon is given by,

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-8}8^x}{x!} \; ; \; x = 0, 1, 2 \ldots$$

The probability that more than 5 calls in between 10 am and 12 noon.

$$P(X>5) = 1 - P(X \le 5)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$$

$$= 1 - \left[\frac{e^{-8}8^0}{0!} + \frac{e^{-8}8^1}{1!} + \frac{e^{-8}8^2}{2!} + \frac{e^{-8}8^3}{3!} + \frac{e^{-8}8^4}{4!} + \frac{e^{-8}8^5}{5!}\right]$$

$$= 1 - [e^{-8}[1 + 8 + 32 + 85.3 + 170.67 + 273.0]]$$

$$= 1 - [e^{-8}(569.97)] = 1 - 0.1912036$$

$$P(X>5) = 0.80879$$

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7) ~~A poisson distribution~~

If X is a poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$. Find

i) λ ii) Mean of X iii) β_1, the coefficient of skewness

If X is a poisson variate with parameter λ.

$$P(X=2) = 9P(X=4) + 90P(X=6) \rightarrow (1)$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; x = 0, 1, 2, \dots$$

From ①

$$P(X=2) = 9\,P(X=4) + 90\,P(X=6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\,\frac{e^{-\lambda}\lambda^4}{4!} + 90\,\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = e^{-\lambda}\left[\frac{9\lambda^4}{24} + \frac{90\lambda^6}{720}\right]$$

$$\frac{e^{-\lambda}\lambda^2}{2} = \frac{e^{-\lambda}\lambda^2}{8}\left[3\lambda^2 + \lambda^4\right]$$

$$1 = \frac{1}{4}(3\lambda^2 + \lambda^4)$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

put $\lambda^2 = y$,

$$y^2 + 3y - 4 = 0$$

$$(y+4)(y-1) = 0$$

$y + 4 = 0$	$y - 1 = 0$
$y = -4$	$y = 1$
$\lambda^2 = y$	$\lambda^2 = 1$
$\lambda^2 = -4$ (Not possible)	$\boxed{\lambda = 1}$

ii) Hence mean $= \lambda$

Mean $= 1$

iii) $\beta_1 = \frac{1}{\lambda} = \frac{1}{1} = 1$

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8) If X and Y are independent ~~Poisson~~ poisson variate such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$. Find the variance of $X - 2Y$.

Soln) $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3) \rightarrow ①$

Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$

We Know that,

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

From (1) $\Rightarrow$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \text{ and } \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-\mu}\mu^3}{3!}$$

$$\lambda e^{-\lambda} = \frac{e^{-\lambda}\lambda^2}{2} \text{ and } \frac{\mu^2 e^{-\mu}}{2} = \frac{\mu^3 e^{-\mu}}{6} \rightarrow (2)$$

Solving eqn (2)

$$\lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2}$$

$$2\lambda e^{-\lambda} = \lambda^2 e^{-\lambda}$$

$$\lambda = 2$$

$$\frac{\mu^2 e^{-\mu}}{2} = \frac{\mu^3 e^{-\mu}}{6}$$

$$3 = \mu$$

$$\mu = 3$$

---

9) Fit a Poisson distribution to the following data

Number of mistakes per page :	0	1	2	3	4	Total
Number of pages :	109	65	22	3	1	200

If the above distribution is approximated by a poisson distribution, then the parameter of poisson distribution is given by.

$$m = \frac{0 \times 109 + 1 \times 65 + 2 \times 22 + 3 \times 3 + 4 \times 1}{200} = 0.61$$

By poisson probability law, the frequency (number) of page containing r mistakes is given by

$$f(r) = N \cdot P(r) = 200 \times \frac{e^{-0.61} (0.61)^r}{r!}$$

Putting $r = 0, 1, 2, \ldots$ we get the expected frequencies of poisson distribution.

[Also, $P(0) = e^{-0.6} = (2.71828)^{-0.61} = \frac{1}{\text{Antilog}(0.61 \times \log 2.71828)}$

$= \frac{1}{\text{Antilog}(0.61 \times 0.4343)} = \frac{1}{\text{Antilog}(0.26492)}$

$= \frac{1}{1.841} = 0.5432.$] ×

Calculation for expected poisson frequencies.

No. of Mistakes		Expected frequency.
0	$f(0) = 200 \times e^{-0.61} = 108.64$	$\simeq 108$
1	$f(1) = 200 \times e^{-0.61} \times 0.61 = 66.27$	$\simeq 66$
2	$f(2) = 200 \times e^{-0.61} \times \frac{(0.61)^2}{2!} = 20.21$	$\simeq 20$
3	$f(3) = 200 \times e^{-0.61} \times \frac{(0.61)^3}{3!} = 4.11$	$\simeq 4$
4.	$f(4) = 200 \times e^{-0.61} \times \frac{(0.61)^4}{4!} = 0.63$	$\simeq 1$

10) After correcting 50 pages of the proof of a book, the proof reader finds that there are, on the average 2 errors per 5 pages. How many pages would one expect to find with 0, 1, 2, 3 and 4 errors, in 1000 pages of the first print of the book?

Let the random variable x denote the number of errors per page. Then the mean number of errors per page is given by $\lambda = \frac{2}{5} = 0.4$

Using poisson probability law, probability of x errors per page is given by

$$P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.4}(0.4)^{x}}{x!} \; ; x = 0, 1, 2, \ldots$$

Expected number of pages with x errors per page in a book of $N = 1000$ pages are.

$$f(x) = N \cdot P(X=x) = 1000 \times \frac{e^{-0.4}(0.4)^{x}}{x!} \; ; x = 0, 1, 2 \ldots$$

using the recurrance.

No. of errors per page (x)	Probability $P(x)$	Expected number of page $f(x) = Np(x)$
0	$p(0) = e^{0.4} = 0.6703$	$670.3 \simeq 670$
1	$p(1) = \frac{0.4}{0+1} p(0) = 0.26812$	$268.12 \simeq 268$
2	$P(2) = \frac{0.4}{1+1} p(1) = 0.053624$	$53.624 \simeq 54$
3	$P(3) = \frac{0.4}{2+1} p(2) = 0.71298$	$7.1298 \simeq 7$
4	$P(4) = \frac{0.4}{3+1} P(3) = 0.0007129$	$0.71298 \simeq 1$

11) Fit a poisson distribution to the following data which gives the number of dodders in a sample of cloves seeds

No. of deeds (x)	0	1	2	3	4	5	6	7
Observed frequency (f)	56	156	132	92	37	22	4	0

$$\text{Mean} = \frac{1}{N} \Sigma f(x) = \frac{986}{500} = 1.972$$

Taking the mean of the poisson distribution we want to fit, we get $\lambda = 1.972$.

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} ; x = 0, 1, 2, \ldots$$

$$P(0) = e^{-\lambda} = e^{-1.972}$$

$$\log_{10} P(0) = -1.972 \log_{10} e = -1.972 \times 0.48424$$

$$= -0.856419 = \bar{1}.143581$$

$$P(0) = \text{Antilog}(\bar{1}.1436) = 0.1392$$

Using the recurrence formula, the various probabilities, viz, $P(1), P(2), \ldots$ can be easily calculated as shown in the following table.

x	$\frac{\lambda}{x+1}$	$p(x)$	Expected frequency $NP(x) = 500 \cdot P(x)$
0	1.972	0.13920	$69.6000 \simeq 70$
1	0.986	0.27455	$137.2512 \simeq 137$
2	0.657	0.27006	$135.3296 \simeq 135$
3	0.493	0.17793	$86.9566 \simeq 89$
4	0.394	0.10964	$43.6556 \simeq 44$

5	0.328	0.03459	17.2966 ≃ 17
6	0.281	0.01137	5.6846 ≃ 6
7	0.247	0.00320	1.6013 ≃ 2
8	0.219	0.00078	0.3942 ≃ 0

Geometric Distribution.

A random variable X is said to have a geometric distribution. If it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = \begin{cases} q^x p & ; x = 0, 1, 2, \ldots, \quad 0 < p \leq 1; q = 1-p \\ 0 & ; \text{otherwise.} \end{cases}$$

Moments of Geometric distribution

$$\text{Mean} = \mu_1' = \sum_{x=0}^{\infty} x \, p(X=x)$$

$$= \sum_{x=1}^{\infty} x . pq^x = pq \sum_{x=1}^{\infty} x q^{x-1}$$

$$\text{Mean} = pq(1-q)^{-2} = \frac{q}{p}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= E[X(X-1)] + E(X) - [E(X)]^2$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \, P(X=x)$$

$$= \sum_{x=2}^{\infty} x(x-1) \, pq^x$$

$$= 2pq^2 \sum_{x=2}^{\infty} \left[\frac{x(x-1)}{2 \times 1} q^{x-2} \right]$$

$$= 2pq^2 (1-q)^{-3} = \frac{2q^2}{p^2}$$

$$\therefore V(X) = \mu_2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p^2}$$

Moment generating function of Geometric Distribution

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} . q^x . p$$

$$= p \sum_{x=0}^{\infty} (e^t q)^x = p(1-qe^t)^{-1} = \frac{p}{1-qe^t}$$

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} p(1-qe^t)^{-1}\right]_{t=0}$$

$$= pq(1-q)^{-2} = q/p$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M(t)\right]_{t=0} = \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\mu_2 = \mu_2' - \mu_1' = \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2}$$

$(q+p)=1$

$$= \frac{q^2 + pq}{p^2} = \frac{q}{p^2}$$

Hence the mean and variance of geometric distribution are $\frac{p}{q}$ and $\frac{q}{p^2}$ respectively.

Remark:

1) $\text{Variance} = \frac{q}{p^2} = \frac{q}{p} \cdot \frac{1}{p} = \frac{\text{Mean}}{p} > \text{mean}$

Hence. For the Geometric distribution variance is greater than the mean.

2) The Probability generating function of the geometric distribution is obtained on replacing e^t by s

$$P_X(s) = \frac{p}{(1-qs)}$$

Practice Questions:

2 Marks:

1. Define Moment Generating Function of poisson distribution
2. Find r^{th} moment x about origin
3. Write the Characteristic function of binomial distribution.
4. Six coins are tossed 6400 times using the Poisson distribution, find the approximate probability of getting 6 heads r times.

5 Marks:

1. If X is said to be a continuous random variable with p.d.f. $f(x) = 6x(1-x), 0 \leq x \leq 1$, Check that f(x) is p.d.f.
2. Effect of change of origin and scale on moment generating function.
3. Explain additive property of Poisson distribution.
4. Derive the Mode of Poisson distribution.
5. A car hire firm has two cars which its fires out day by day.then the number of demand on each day is distributed has poisson variate with mean 1.5, calculate the preposition of days of which (i) Neither car is used (ii) some demand is refused.

10 Marks:

1. Derive Reproductive property of independent of Poisson distribution.
2. Explain fitting of Poisson distribution.
3. After orreting 50 pages of the proof of a book, the proof reader finds that there are on the average of 2 errors per 5 pages. How many pages would are expect to find out with 0,1,2,3 and 4 errors in 1000 pages of the first print of the book.

References:

S.C. Gupta and V.K. Kapoor, Fundamentals of Mathematical Statistics, Sultan Chand &Co, New Delhi, Reprint 2019.