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**II M.Sc Mathematics – Semester - III**

**E-Notes (Study Material)**

| **Core Course : Topology** **Code: 23PMA33** |
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| **UNIT-V:** Countability and Separation Axiom: The Countability Axioms – The separation Axioms – Normal spaces – The Urysohn Lemma – The Urysohnmetrization Theorem – The Tietz extension theorem. |
| **Learning Objectives:** To study Countability and Separation Axiom: The Countability Axioms – The separation Axioms – Normal spaces – The Urysohn Lemma – The Urysohnmetrization Theorem – The Tietz extension theorem. |
| **Course Outcom**e: Understanding first and second countability, which help in defining sequences and bases in topology. Learning different levels of separation (T0, T1, T2, etc.) to distinguish points and sets in a space. Exploring spaces where disjoint closed sets can be separated by disjoint open sets. Understanding how a continuous function can separate closed sets in a normal space. Studying conditions under which a topological space can be given a metric. Learning how a continuous function defined on a closed subset can extend to the whole space. |

**Overview:**

**Countability Axioms** – These define conditions for a space to have a countable base (first and second countability).

**Separation Axioms** – These describe how well a space separates points and sets (T0, T1, T2, etc.).

**Normal Spaces** – Spaces where two disjoint closed sets can be separated by disjoint open sets.

**Urysohn Lemma** – States that in a normal space, a continuous function can separate two closed sets.

**Urysohn Metrization Theorem** – Provides conditions under which a topological space can be turned into a metric space.

**Tietze Extension Theorem** – Ensures that a continuous function defined on a closed subset can extend to the whole space.

These concepts help classify spaces and understand their properties in topology.

UNIT-V

# COUNTABILITY AND SEPARATION AXIOM

## Sec 30: The Countability Axioms.

Def : 1st Countability Axiom :

A Space X is said to have a countable basis at $x$ if there is a countable collection $\mathcal{B}$ of nbd of $x$ such that each nbd of $x$ contains atleast One of the element of $\mathcal{B}$.

A Space that has a countable basis at each of its pts is said to satisfy the 1st Countability axiom or to be 1st Countable.

Thm: 30.1.

Let X be a topological Space.

a) Let A be a subset of X. If there is a seq of pts (Sequence) of A cgs (Converging) to $x$ then a $x \in \bar{A}$, the converse holds if X is 1st Countable.

b) Let $f: X \to Y$. If $f$ is cts then for every cgt seq $x_n \to x$ in Y, then seq $f(x_n) \to f(x)$. The converse holds if X is 1st Countable.

Proof:

(a): Let A be a subset of X.

If $x_n \to x$ where $x_n \in A$ then every nbd U of $x$ contains a pt of A.

By thm, "Let A be a subset of the topological space Then $x \in \bar{A}$ iff every open set U containing $x$ intersects A".

$\therefore x \in \overline{A}$

conversely,

Let $X$ be a $1^{st}$ Countable Space & $A \subset X$.

Let $x \in \overline{A}$, we have to find a sequence $\{x_n\}$ in $A$ converging to $x$.

$\because x$ is $1^{st}$ Countable $\exists$ a Countable basis at $x$.

Let it be $B_x$

$\therefore \mathcal{B}_x = \{B_n / n \in Z_+$ & $B_n$ is a nbd of $x\}$

Let $C_1 = B_1$

$C_2 = B_1 \cap B_2$

$\vdots$

$C_n = B_1 \cap B_2 \cap \dots \cap B_n = \bigcap_{i=1}^{n} B_i$

$\therefore C_1, C_2 \dots$ are nbd of $x$.

Also $C_1 \supset C_2 \supset C_3 \supset \dots \supset C_{n_0} \supset \dots$

Let $\mathcal{B}' = \{C_n / n \in Z_+\}$

Then $\mathcal{B}'$ is countable and each $C_n$ is non empty.

Also $\mathcal{B}'$ is a basis at $x$.

For let $U$ be any open set containing $x$.

$\therefore \exists$ a basis element $B_n \in \mathcal{B}_x$ such that

$x \in B_m \subset U$.

Also $B_m \supset C_m$

$\Rightarrow x \in C_m \subset U$, where $C_m \in \mathcal{B}$.

Given $x \in \overline{A}$

$\therefore C_n \cap A \neq \phi, \forall n$.

Let $x_n \in C_n \cap A, \forall n$.

Then $(x_n)$ is the required seq of pts of $A$.

CLAIM :- $(x_n) \to x$

let $V$ be any nbd of $x$.

$\because \mathcal{B}'$ is a basis at $x$, $\exists!$ $n_0 \in \mathbb{Z}_+$ such that

$x \in C_{n_0} \subset V$.

But $C_m \subset C_{n_0}$ ; $\forall\ m \geq n_0$

$\therefore x_m \in C_{n_0}$ ; $\forall\ m \geq n_0$

$x_m \in V$, $\forall\ m \geq n_0$.

$\therefore (x_n) \to x$.

b): Assume that $f$ is cts.

Gn: $x_n \to x$.

T.P: $f(x_n) \to f(x)$.

let $V$ be a nbd of $f(x)$.

Then $f^{-1}(V)$ is a nbd of $x$ ($\because f$ is cts)

$\because x_n \to x$, there is an $N$ such that

$x_n \in f^{-1}(V)$ for $n \geq N$

$\Rightarrow f(x_n) \in V$ for $n \geq N$

$\therefore f(x_n) \to f(x)$.

Conversely. Gn: $(x_n) \to x$ in $X$

$\Rightarrow f(x_n) \to f(x)$ in $Y$

T.P: $f: X \to Y$ is cts.

let $A$ be a subset of $X$.

It is enough to P.T $f(\bar{A}) \subset \overline{f(A)}$.

Let $f(x) \in f(\bar{A})$

$\therefore x \in \bar{A}$

$\therefore x$ ... , applying the converse part of result (a)

$\therefore \exists$ a seq $(x_n)$ in $A$ converging to $x$.

By hypothesis $f(x_n) \to f(x)$ in $f(A)$.

Again by result (a).

$\Rightarrow f(x) \in \overline{f(A)}$

$\therefore f(\bar{A}) \subset \overline{f(A)}$

$\Rightarrow f: X \to Y$ is cts.

Def: 2nd Countability Axiom:

If a space $X$ has a countable basis for its topology then $X$ is said to satisfy the Second Countability axiom or to be 2nd Countable.

Thm 30.2.

A Subspace of a 1st Countable space is 1st countable and a countable product of 1st Countable space is 1st countable. A subspace of a 2nd countable space is 2nd countable and a countable product of 2nd countable space is 2nd Countable.

Proof:

T.P: A subspace of a 1st countable space is 1st countable

Let $X$ be a 1st countable space.

Let $Y$ be a subspace of $X$.

T.P: $Y$ is 1st Countable.

Let $y \in Y$, then $y \in X$.

$\because X$ is 1st Countable $\exists$ a Countable basis at $y$ in $X$.

let it be $\mathcal{B}$ where

$\mathcal{B} = \{B_n / n \in Z_+$ and each $B_n$ is a nbd of $y$ in $X\}$.

Let $\mathcal{B}_Y = \{B_n \cap Y / n \in Z_+$ and $B_n \in \mathcal{B}\}$

Clearly $\mathcal{B}_Y$ is countable.

CLAIM:

$\mathcal{B}_Y$ is a countable basis at $y$ in $Y$.

Let $V$ be any open set in $Y$ containing $y$.

$\therefore V = U \cap Y$ where $U$ is open in $X$.

$\therefore y \in U$ and $U$ is open in $X$ and $\mathcal{B}$ is a basis at $y$ in $X$. countable basis

$\therefore \exists\ B_m \in \mathcal{B}$ such that $y \in B_m \subset U$.

$\therefore y \in B_m \cap Y \subset V$.

$\therefore \mathcal{B}_Y$ is a Countable basis at $y$ in $Y$.

$\therefore Y$ has a countable basis element at each of its points. $\therefore Y$ is $1^{st}$ Countable.

T.P: A countable product of $1^{st}$ Countable Space is $1^{st}$ Countable.

Let $\{X_i\}_{i=1}^{\infty}$ be a countable collection of $1^{st}$ Countable Space.

T.P: $\prod_{i=1}^{n} X_i$ is $1^{st}$ Countable.

Let $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i$, then $x_i \in X_i, \forall i$

$\because X_i$ is $1^{st}$ countable $\exists$ a Countable basis $\mathcal{B}_{x_i}$ at $x_i$ in $X_i$.

$\therefore \mathcal{B}_{x_i} = \{B_{x_{i,j}} / B_{x_{i,j}}$ is a nbd of $x_i$ in $X_i$ & $j = 1, \ldots \infty\}$

Let $\mathcal{B}_x = \{\prod_{j=1}^{\infty} B_{x_{i,j}} / B_{x_{i,j}}$ is a nbd of $x_i$ in $X_i$, $i = 1, \ldots \infty\}$

CLAIM:

$\mathcal{B}_x$ is a countable basis of $x$ in $\prod_{i=1}^{n} X_i$

Let $U$ be any open set in $\prod_{i=1}^{\infty} X_i$ containing $x$.

$\therefore U = \prod_{i=1}^{\infty} U_i$, where $U_i = U_i$ for $i = 1, 2 \ldots n$.

$= X_i$ for $i \neq 1, 2 \ldots n$.

$\therefore x_i \in U_i$ if $i = 1, 2, \ldots, n$.

(or) $x_i \in X_i$ if $i \neq 1, 2 \ldots n$.

Then $\because \mathcal{B}_{x_i}$ is a countable basis at $x_i$ in $X_i$.

$\exists B_{x_{i,k}} \in \mathcal{B}_{x_i}$ such that $x_i \in \mathcal{B}_{x_{i,k}} \subset U$, $i = 1, 2 \ldots n$

for some $k$.

$\therefore x \in \prod B_{x_{i,k}} \subset \prod U_i$

i.e., $x \in \prod B_{x_{i,k}} \subset U$.

$\therefore \mathcal{B}_x$ is a countable basis at $x$.

$\therefore \prod X_i$ is 1st countable.

$y \in U$ and $U$ is open in $X$ and $\mathcal{B}$ is a basis at $y$ in $X$.

$\therefore \exists B_m \in \mathcal{B}$ such that $y \in B_m \subset U$.

$\therefore y \in B_m \cap Y \subset U \cap Y$

i.e., $Y \in B_m \cap Y \subset U$.

$\therefore \mathcal{B}_Y$ is a countable basis at $y$ in $Y$.

$\therefore Y$ has a countable basis element at each of its points.

T.P: A subspace of a $2^{nd}$ countable space is $2^{nd}$ countable

Let $Y$ be a subspace of $X$.

T.P: $Y$ is $2^{nd}$ Countable.

$\because$ $X$ is $2^{nd}$ Countable.

$X$ has a countable basis for its topology.

Let it be $\mathcal{B} = \{B_n \cap Y / n \in \mathbb{Z}_+$ and $B_n \in \mathcal{B}\}$

$\therefore$ $\mathcal{B}_Y$ is countable.

CLAIM:

$\mathcal{B}_Y$ is a basis for the topology on $Y$.

Let $y \in Y$ and let $V$ be an open set in $Y$ containing $y$.

$\therefore$ $V = U \cap V$ where $U$ is open in $Y$.

$\therefore$ $y \in U$ and ~~wh~~ $U$ is open in $X$ and $\mathcal{B}$ is a basis

for the topology of $X$.

$\therefore$ $\exists$ $B_m \in \mathcal{B}$ such that $y \in B_m \subset U$.

$\therefore$ $y \in B_m \cap Y \subset U \cap Y$.

i.e., $y \in B_m \cap Y \subset V$.

$\therefore$ $\mathcal{B}_Y$ is a basis for the topology on $Y$.

$\therefore$ $Y$ is $2^{nd}$ countable.

T.P: A countable product of $2^{nd}$ Countable Space

is $2^{nd}$ countable.

Let $\{X_i\}_{i=1}^{\infty}$ be countable collection of $2^{nd}$

countable Space

T.P: $\prod_{i=1}^{\infty} X_i$ is $2^{nd}$ countable.

$\because$ each $X_i$ is $2^{nd}$ countable.

$\exists$ a countable basis $\mathcal{B}_i$ for the topology on $X_i$

where $\mathcal{B} = \{B_{i,j} / j = 1, 2 \ldots \infty$ & $B_{i,j}$ is an open set in $X_i\}$

Let $\mathcal{B} = \left\{ \prod_{i=1}^{\infty} B_{i,j} / B_{i,j} \in \mathcal{B}_i, \mathcal{B}_{i,j} \neq X_i \right.$ for finitely many

Clearly $\mathcal{B}$ is countable.

CLAIM:

$\mathcal{B}$ is a basis for the topology on $\prod X_i$.

Let $x = (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i$

Let $\prod_{i=1}^{\infty} U_i$ be any open set containing $x$ then

$U_i = U_i$ for $i = 1, 2 \ldots n$

$= X_i$ for $i \neq 1, 2 \ldots n$

$\therefore x_i \in U_i$ for $i = 1, 2 \ldots n$ (or)

$x_i \in X_i$ for $i \neq 1, 2 \ldots n$.

$\because \mathcal{B}_i$ is the basis for $X_i$, $\exists$ $B_{i,j} \in \mathcal{B}_i$ such that

$x_i \in B_{i,j} \subset U_i$ for some $j$.

Let $W = \prod W_i$ where $W_i = B_{i,j}$ if $i = 1, 2 \ldots n$.

$= X_i$ if $i \neq 1, 2 \ldots n$.

Then $W$ is a nbd of $x$ and $x \in W \subset \prod U_i$

where $W \in \mathcal{B}$.

$\therefore \mathcal{B}$ is a basis for the topology on $\prod X_i$.

$\therefore \prod X_i$ is a $2^{nd}$ countable.

Result :-

Let $X$ be a second countable space is $1^{st}$ countable but converse is not true.

i.e. $2^{nd}$ Countable $\Rightarrow$ $1^{st}$ Countable.

But $1^{st}$ Countable $\not\Rightarrow$ $2^{nd}$ Countable.

Proof:

Let $X$ be a second countable space.

T.P : $X$ is $1^{st}$ countable.

i.e. T.P: $X$ has a countable at each of its pts.

Let $x \in X$.

$\because$ $X$ is second countable, $X$ has a countable basis for its topology.

Let it be $\mathcal{B} = \{B_n / n \in Z_+$ and $B_n$ is open in $X\}$

Consider, $\mathcal{B}_x = \{B_{x,i} / B_{x,i}$ is a nbd of $x$ and $B_{x,i} \in \mathcal{B}\}$

then $\mathcal{B}_x$ is countable ($\because$ $\mathcal{B}$ is countable).

CLAIM:

$\mathcal{B}_x$ is a basis at $x$.

Let $U$ be any open set containing $x$.

$\because$ $\mathcal{B}$ is a basis for the topology on $X$ $\exists$ a basis element $B_m \in \mathcal{B}$ $\quad x \in B_m \subset U$.

Now $B_m \in \mathcal{B}$ and $B_m$ is a nbd of $x$.

$\therefore$ $B_m \in \mathcal{B}_x$.

$\therefore$ $\mathcal{B}_x$ is a basis at $x$.

$\therefore$ $X$ is $1^{st}$ countable.

But the converse is not true.

But the converse is not true. But any $1^{st}$ Countable space need not be $2^{nd}$ Countable.

i.e. any $1^{st}$ Countable space

$R_l$. Real line

T.P: $R_l$ is $1^{st}$ Countable.

Let $x \in R_l$

Let $\mathcal{B}_x = \{ [x, x+1/n] / n \in Z_+ \}$

Then $\mathcal{B}_x$ is a countable basis at $x$ $\forall$ $x \in X$.

$\therefore R_l$ is $1^{st}$ countable.

Now we shall show that $R_l$ has no countable basis for this topology.

Suppose $\exists$ a countable basis $\mathcal{B}$ for its topology

choose for each $x$,

A basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x+1)$

If $x \neq y$ then $B_x \neq B_y$.

Thus for each $x \in R_l$ $\exists$ a basis element $B_x \in \mathcal{B}$.

$\because R_l$ is uncountable, $\mathcal{B}$ is uncountable.

Which is a contradiction to our assumption

$\therefore R_l$ has no countable basis.

$\therefore R_l$ is not $2^{nd}$ countable.

Def: Dense

A subset $A$ of a space $X$ is said to be dense in $X$ if $\bar{A} = X$.

Thm: 30.3

Suppose that $X$ has a countable basis then,

a) Every open covering of $X$ contains a countable subcollection covering $X$.

b) $\exists$ a countable subset of $X$ that is dense in $X$

Proof:

Given that $X$ has a countable basis.

Let $\mathcal{B} = \{B_n / n \in Z_+\}$ be the countable basis for the topology on $X$.

Let $A$ be an arbitrary open covering of $X$

We have to find a countable subcover for $A$.

Let $x \in X$.

$\because$ $\mathcal{A}$ is an open cover, $\exists$ an open set $A \in \mathcal{A}$ such that $x \in A$.

$\because$ $\mathcal{B}$ is a basis $\exists$ a basis element $B_n \in \mathcal{B}$ such that $x \in B_n \subset A$.

Thus for each $n \in \mathbb{Z}_+$ and for $B_n \in \mathcal{B}$ $\exists$ $A \in \mathcal{A}$ such that $B_n \subset A$.

Denote this $A$ by $A_n$

Let $\mathcal{A}' = \{A_n / A_n \in \mathcal{A} \text{ and } B_n \subset A_n\}$

clearly $\mathcal{A}'$ is a countable and it is a subcollection of $\mathcal{A}$.

CLAIM:

$\mathcal{A}'$ is a covering for $X$.

Let $y \in X$.

$\because$ $\mathcal{B}$ is a basis $\exists$ $B_m \in \mathcal{B}$ such that $y \in B_m$

corresponding to this $B_m$, $\exists$ $A_m \in \mathcal{A}$ such that

$B_m \subset A_m$

$\therefore y \in B_m \subset A_m$

$\therefore y \in A_m$

$\therefore$ $\mathcal{A}'$ is a covering for $X$.

Thus $\mathcal{A}$ has a countable subcover $\mathcal{A}'$ covering $X$.

Hence result (a)

T.P: (b)

Let $X$ has a countable sub basis

T.P: $\exists$ a countable dense subset for $X$.

Let $\mathcal{B} = \{B_n / n \in \mathbb{Z}_+\}$ be the countable basis for each $n \in \mathbb{Z}_+$, choose $x_n \in B_n$

Let $A = \{x_n / x_n \in B_n \text{ and } B_n \in \mathcal{B}\}$

Then $A$ is countable.

i.e. T.P: $\bar{A} = X$.

Let $U$ be any nbd $x$ where $x \in X$.

$\because \mathcal{B}$ is a basis $\exists$ a basis element $B_n \in \mathcal{B}$ such that $x \in B_n \subset U$.

But $x_n \in U$ ($\because x_n \in B_n$)

Also $x_n \in A$.

$\therefore U \cap A \neq \phi$ where $U$ is a nbd of $x$.

$\therefore x \in \bar{A}$

$\therefore x \in X \Rightarrow x \in \bar{A}$

$\therefore \bar{A} = X$

$\therefore A$ is dense in $X$

Def:

A space having a countable dense subset is said to be seperable.

Def: Lindelof:

A space for which every open covering contains a countable subcovering is called Lindelof space.

Note:

i) 2nd Countable $\Rightarrow$ Lindelof

ii) 2nd Countable $\Rightarrow$ Separable

iii) 2nd Countable $\Rightarrow$ 1st Countable.

Eg:

1) The space $R_l$ satisfies all the countability axioms but the second.

Soln:

$R_l$ is Separable.

$\because \bar{Q} = R = R_l$, $Q$ is the required countable dense subset of $R_l$.

$\therefore R_l$ is separable.

To show that $R_l$ is Lindelof.

Let $A = \{[a_\alpha, b_\alpha] / \alpha \in J$ and $a_\alpha, b_\alpha \in \mathbb{R}\}$ be an open cover for $\mathbb{R}_l$.

We have to find a countable subcover for a $A$ covering $\mathbb{R}_l$.

Let $C = \bigcup (a_\alpha, b_\alpha)$

$\alpha \in J$ & $[a_\alpha, b_\alpha] \in A$

Then $C$ is a subspace of $\mathbb{R}$.

$\therefore C$ is a $2^{nd}$ Countable ($\because \mathbb{R}$ is $2^{nd}$ countable).

[Subspace of a $2^{nd}$ countable space is $2^{nd}$ countable]

$\therefore C$ is Lindelöf ($\because 2^{nd}$ Countable $\Rightarrow$ Lindelöf).

$\therefore \exists$ a countable subcover $\{(a_{\alpha_i}, b_{\alpha_i}) \; (i = 1, 2 \dots \infty)\}$ covering $C$.

$\therefore \{[a_{\alpha_i}, b_{\alpha_i}) / i = 1, 2 \dots \infty\}$ is countable subcovering for $C$ in $\mathbb{R}_l$ $\longrightarrow$ (1)

Next we shall show that $\mathbb{R} - C$ is countable.

Let $x \in \mathbb{R} - C$ then $x = a_\beta$ for some $\beta \in J$.

Choose a rational no. $q_x \in (a_\beta, b_\beta)$

$\because (a_\beta, b_\beta) \subset C$, $(a_\beta, q_x)$ & $(q_x, b_\beta)$ are also contained in $C$ and $(a_\beta, q_x) = (x, q_x)$.

It follows that if $x$ and $y \in \mathbb{R} - C$ with $x < y$ then $q_x < q_y$.

$\therefore$ For each $x \in \mathbb{R} - C$ we can find a rational $q_x$.

$\therefore$ The map $\mathbb{R} - C \rightarrow \mathbb{Q}$ is 1-1

$\therefore \mathbb{R} - C$ is countable.

For each $x \in \mathbb{R} - C$ we can find an open set from $A$ containing $x$.

$\therefore$ For $\mathbb{R} - C$ $\exists$ a countable subcover from $A$ $\to$ (2)

From ① & ②

$R_l$ has a countable subcover.

$\therefore R_l$ is lindelof.

UQ ⑧ 27 The product of 2 lindelof space need not be lindelof

The space $R_l$ is lindelof.

We shall prove that $R_l^2$ is not lindelof.

The basis of $R_l^2$ is $\mathcal{B}$ where,

$\mathcal{B} = \{[a,b) \times [c,d) / a,b,c,d \in \mathbb{R}\}$

Consider L to be the $\{(x,-x) / x \in \mathbb{R}\}$ then

L is a subspace of $\mathbb{R}_l^2$.

L is closed in $\mathbb{R}_l^2$.

($\because$ L is the diagonal of the hausdroff space $\mathbb{R}_l^2$)

Consider $(\mathbb{R}_l^2 - L) \cup \{[a,b) \times [-a,d) / a,b,d \in \mathbb{R}\} \rightarrow (*)$

which is an open cover for $\mathbb{R}_l^2$.

All basis element of the form $[a,b) \times [-a,d)$ intersect L in atmost one pt namely $(a,-a)$.

$\because$ L is uncountable.

No countable subcollection covers L.

Here there is no countable subcover.

for the cover given in (*) & covering $\mathbb{R}_l^2$.

$\therefore \mathbb{R}_l^2$ is not lindelof

Note:

Subspace of a lindelof space need not be lindelof.

Eg: The space $I \times I$ under dictionary order is called an ordered square and is denoted by $I_o^2$ where $I = [0,1]$.

soln :

$\because I_o^2$ is compact, $I_o^2$ is lindelof.

But $A = I \times [0,1)$ is not lindelof for $A = \bigcup U_x$ where $U_x = \{x\} \times [0,1)$ which is open in A.

$\{U_x / x \in I\}$ is uncountable and no proper subcollection covers A.

Sec 31 : The Separation Axioms

Regular Space :

Let X be a space with one pt sets are closed in X. Then X is said to be regular if for each pair consisting of a pt $x$ and a closed set B disjoint from $x$ $\exists$ disjoint open sets containing $x$ and B respectively.

Normal Space :

Let X be a space with one pt sets are closed in X. Then X is said to be normal, if A and B are disjoint closed sets in X then $\exists$ disjoint open sets containing A and B respectively.

Thm:

1. Every regular space is Hausdorff space

Proof :

Let X be a regular space.

T.P : X is Hausdorff.

Let $x, y \in X$ such that $x \neq y$.

$\because$ X is regular and $y \in X$,

We have $\{y\}$ is closed in X

$\because y \neq x$, $x \notin \{y\}$.

i.e., $x \in X$ and $\{y\}$ is a closed set in X not containing $x$ and X is regular.

$\Rightarrow \exists$ disjoint open sets $U$ and $V$ such that $x \in U$ and $\{y\} \subseteq V$ i.e. $y \in V$.

Thus we have found out two disjoint open set $U$ & $V$ such that $x \in U$ and $y \in V$.

$\therefore X$ is Hausdorff space.

Note:

But the converse is not true.

i.e. Hausdorff space need not be regular.

2) Every normal space is regular.

Proof.

Let $X$ be a normal space.

T.P: $X$ is regular.

i.e. To prove for every pair consisting of a pt $x$ & a closed set not containing $x$.

$\exists$ disjoint open set containing $x$ & $B$ respectively.

Let $x \in X$ and let $B$ be a closed set in $X$ not containing $x$.

Now $\{x\}$ is a closed set in $X$ ($\because X$ is normal)

Now $\{x\}$ and $B$ are 2 disjoint closed sets.

$\because X$ is normal $\exists$ disjoint open sets $U$ and $V$ such that $\{x\} \subseteq U$ and $B \subseteq V$.

$\therefore x \in U$ and $B \subseteq V$.

$\therefore X$ is regular.

Note: But the converse is not true.

i.e. Regular space need not be normal space.

Proof: The three seperation axioms are illustrated in below.

Lemma 31.1

Let $X$ be a topological space. Let one point sets in $X$ be closed.

a). $X$ is regular $\iff$ given a pt $x$ of $X$ and a nbd $U$ of $x$ there is a nbd $V$ of $x$ such that $\bar{V} \subset U$.

b). $X$ is normal $\iff$ given a closed set $A$ and an open set $U$ containing $A$, there is a open set $V$ containing $A$ such that $\bar{V} \subset U$.

Proof:

a). Let $X$ be regular.

$\Rightarrow$ one pt sets are closed.

Let $x \in X$ and $U$ be an open set s.t $x \in U$

$\therefore x \notin X - U$.

$\because U$ is an open set, $X - U$ is a closed sets not containing $x$.

Let $B = X - U$ which is closed set not containing $x$.

Now $x \in X$ and $B$ is a closed set not containing $x$.

$\because X$ is regular. $\exists$ a disjoint open sets $V$ containing $x$ and $W$ containing $B$ $\longrightarrow$ (1).

CLAIM: $\bar{V} \subset U$.

$V = X - W$.

Now $V \cap W = \phi$ (by (1))

$\Rightarrow V \subset X - W$ and $X - W$ is closed.

($\because W$ is an open set containing $B$).

$\Rightarrow \bar{V} \subset X - W$ ($\because \bar{V}$ is the smallest closed set containing $V$)

Now $B \subset W \rightarrow$ (2)

$\Rightarrow X - W \subset X - B$

We have $\bar{V} \subset X - W$ and $X - W \subset X - B$

i.e> $\bar{V} \subset X - W \subset X - B$

i.e> $\bar{V} \subset X - B$

i.e> $\bar{V} \subset U$.

$\therefore \exists$ a open set $V$ of $x$. Such that $x \in V$ and $\bar{V} \subset U$

conversely,

Suppose one point sets are closed and $U$ is a Open set containing $x$ then $\exists$ a nbd $V$ of $x$ Such that, $x \in V$ & $\bar{V} \subset U$.

T.P :.

$X$ is regular.

We have one pt are closed.

let $x \in X$ and $B$ be a closed set not containing

let $U = X - B$

$\Rightarrow U$ is an open set containing $x$.

$\therefore \exists$ an open set $V$. st $x \in V$ & $\bar{V} \subset U$.

(By hypothesis).

Now $V$ and $X - \bar{V}$ are disjoint open sets S.t $x \in V$ & $B \subseteq X - \bar{V}$.

$(\because \bar{V} \subset U \Rightarrow X - U \subset X - \bar{V} \Rightarrow B \subseteq X - \bar{V})$

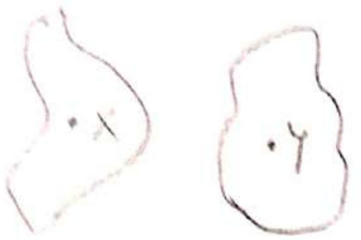
Hence $X$ is regular.

T.p : (b).

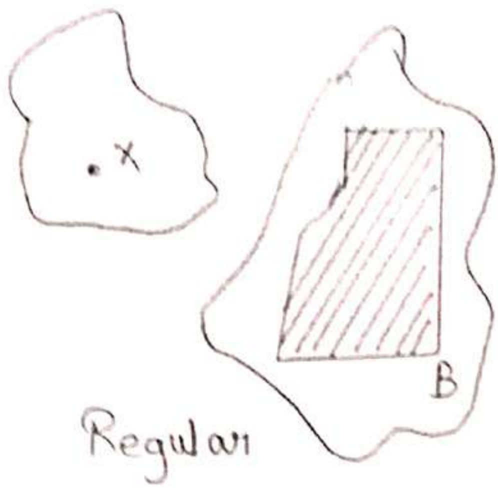
Let $X$ be normal.

$\Rightarrow$ one pt sets are closed.

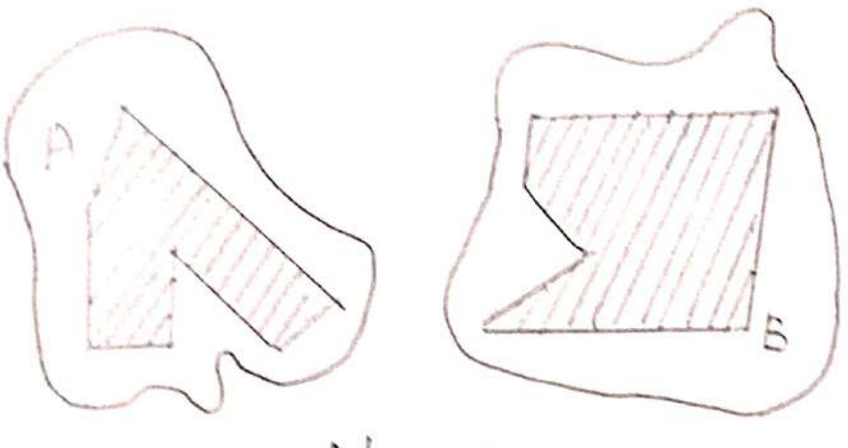
Let $A$ be a closed set in $X$ & $U$ be an open set containing $A$




Hausdorff




Regular




Normal.

$\therefore A \subseteq U$.

$\because X$ is normal.

If $A$ and $B$ are disjoint closed sets in $X$ then $\exists$ disjoint open set $V$ containing $A$ and $W$ containing $B$ respectively. where $B = X - U$.

Claim: $\bar{V} \subset U$.

Now $V \cap W = \phi$

$\Rightarrow V \subset X - W$ and $X - W$ is closed.

$\Rightarrow \bar{V} \subset X - W$ [ $\bar{V}$ is the smallest closed set containing $V$ ].

Now $B \subset W$

$\Rightarrow X - W \subset X - B$

We have, $\bar{V} \subset X - W$ & $X - W \subset X - B$.

i.e., $\bar{V} \subset X - W \subset X - B$

i.e., $\bar{V} \subset X - B$

$\Rightarrow \bar{V} \subset U$.

Conversely,

Suppose given a closed set $A$, $U$ is an open set containing $A$ then $\exists$ a nbd $V$ of $A$ such that $A \subseteq V$ and $\bar{V} \subset U$.

T.P: $X$ is normal.

We have one pt sets are closed.

Let $A$ & $B$ are two disjoint closed set in $X$.

let $U = X - B$.

$\Rightarrow U$ is an open set containing $A$.

$\Rightarrow \exists$ an open set $V$. s.t $A \subseteq V$ and $\bar{V} \subset U$ (by hypothesis).

$\therefore X - U \subseteq X - \bar{V}$

i.e. $B \subseteq X - \bar{V}$. which is open

$\therefore V$ & $X - \bar{V}$ are the required disjoint open sets containing $A$ & $B$ respectively.

$\therefore X$ is normal.

Thm 31.2.

a). A subspace of a H. S (Hausdroff space) is Hausdroff, a product of H.S is Hausdroff.

b). A subspace of a regular space is regular, a product of a regular space is regular.

T.P: i). Subspace of a regular Space is regular.

Let $X$ be a regular space.

Let $Y$ be a subspace of $X$.

T.P: $Y$ is regular.

$\because X$ is regular, $X$ is hausdroff & and hence one pt sets are closed in $Y$.

Let $y \in Y$.

$\therefore y \in X$.

$\therefore \{y\}$ is closed in $X$.

$\therefore \{y\} \cap Y$ is closed in $Y$.

but $\{y\} \cap Y = \{y\}$.

$\therefore \{y\}$ is closed in $Y$.

$\therefore$ one pt sets are closed in $Y$.

let $x \in Y$ and $B$ be a closed set in $Y$.

s.t $x \notin B$.

$x \in Y \Rightarrow x \in X \quad (\because Y \subset X)$.

B is a closed set in $Y \Rightarrow B = \bar{B} \cap Y$. where $\bar{B}$ is a closure of B in X.

$x \notin B \Rightarrow x \notin \bar{B}$ where $\bar{B}$ is a closed set in X.

$\because$ X is regular, $\exists$ open sets U & V in X.

s.t $x \in U$ & $\bar{B} \subseteq V$ & $U \cap V = \phi$.

i.e., $U \cap V$ and $V \cap Y$ are two disjoint open set in Y

Such that $x \in U \cap Y$ and $B \subseteq V \cap Y$.

$\therefore$ Y is regular.

(ii) T.P: product of a regular space is regular.

Let $X = \Pi X_\alpha$. where each $X_\alpha$ is regular.

T.P: X is regular.

Let $x \in X$.

Let $x = (x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha \Rightarrow x_\alpha \in X_\alpha \; \forall \alpha \in J$.

Let U be a nbd of x in X.

$\exists$ a basis element $\Pi U_\alpha$; where $U_\alpha$ is open in $X_\alpha \; \forall \alpha$ and $U_\alpha = X_\alpha$ except for finitely many values of $\alpha$ and $\Pi U_\alpha \subseteq U$.

$\therefore (x_\alpha) \in \Pi U_\alpha$.

$\Rightarrow x_\alpha \in U_\alpha \; \forall \alpha$.

i.e., $x_\alpha \in U_\alpha$ and $U_\alpha$ is an open set containing $x_\alpha$ and $X_\alpha$ is regular.

$\Rightarrow \exists$ a nbd $V_\alpha$ of $x_\alpha$ in $X_\alpha$ s.t $\overline{V_\alpha} \subseteq U_\alpha$.

(31.1) $\therefore \overline{V_\alpha} \subseteq U_\alpha$ for finitely many values of $\alpha$.

$\Rightarrow \prod \bar{V}_\alpha \subseteq \prod U_\alpha$

$\Rightarrow \prod \bar{V}_\alpha \subseteq \prod U_\alpha \subseteq U$ $\quad (\because \prod \bar{A}_\alpha = \overline{\prod A_\alpha})$

let $V = \prod V_\alpha$ where $V_\alpha$ is an open set containing $x_\alpha$ for finitely many values of $\alpha$ and Hence $V$ is a basic open set containing $x$.

i.e., $x \in V$.

i.e., $V$ is a nbd of $x$ s.t $\bar{V} \subseteq U$ and hence $X$ is regular.

$\therefore \prod X_\alpha$ is regular.

Eg:

The Space $\mathbb{R}_1$ is normal

let $A, B$ be two disjoint closed sets in $\mathbb{R}_1$.

Let $a \in A$

$\Rightarrow a \in \mathbb{R}_1 - B$ and $\mathbb{R}_1$ is open.

$\Rightarrow \exists$ a basic open set $[a, x_a]$ such that $a \in [a, x_a) \subseteq \mathbb{R}_1 - B$.

i.e., For each $a \in A$, we can find a basic open set $[a, x_a)$ which is disjoint from $B$.

i.e., $[a, x_a) \cap B = \phi \longrightarrow (1)$

III$^{ly}$, For each $b \in B$. We can. Find a basic open set $[b, x_b)$ which is disjoint from $A$.

i.e., $[b, x_b) \cap A = \phi \longrightarrow (2)$

Let $= \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$

clearly $U$ and $V$ are open sets s.t $A \subseteq U$ & $B \subseteq V$.

Claim:

$U \cap V = \phi$ Suppose $U \cap V \neq \phi$

Let $x \in U \cap V \Rightarrow x \in U$ and $x \in V$.

$\Rightarrow x \in [a, x_a)$ and $x \in [b, x_b)$ for some $a \in A, b \in B$

If $x \in [a, x_a) \Rightarrow x \notin B$.

$x \in [b, x_b) \Rightarrow x \notin A$.

$x \in [a, x_a) \cap [b, x_b)$.

$\therefore$ Without the loss of generality. We assume that

$a < b$. $\therefore a < b < x < x_a$

$\Rightarrow b \in [a, x_a)$

Which is a $\Rightarrow\Leftarrow$ to (1) ($\because b \in B$, $[a, x_a) \cap B = \phi$)

i.e., $U \cap V = \phi$

$\therefore R_l$ is normal.

Sec 32: Normal Spaces.

Thm: 32.1

Every regular Space with a countable basis is normal.

Let $X$ be a regular Space with a countable basis $\mathcal{B}$.

$\because X$ is regular, singleton sets are closed in $X$.

Let $A$ and $B$ be two disjoint non-empty closed Subsets of $X$.

Let $x \in A$

$\Rightarrow x \notin B$ ($\because A \cap B = \phi$)

$\Rightarrow x \in X - B$ which is open ($\because B$ is closed).

$\Rightarrow \exists$ a nbd $V$ of $x$, s.t $x \in X$ and $\bar{V} \subset X - B$

$\because$ X is regular and by thm [31.1 (a)]

i.e> $x \in V$ & V is open.

$\Rightarrow \exists$ a basis open set U of $\mathcal{B}$. Such that

$x \in U \subset V$ and $\bar{U} \subset \bar{V} \subset X - B$.

$\Rightarrow \bar{U} \cap B = \phi \quad (\because \bar{U} \subset X - B)$.

i.e> For each $x \in A$ $\exists$ a basis open set, $U \in \mathcal{B}$ s.t $x \in U$ & $\bar{U}$ doesn't interest B.

i.e. $\exists$ a countable open cover $\{U_n\}$ for A with each $\bar{U}_n$ not intersecting A.

$|||^{ly}$, $\exists$ a countable open cover $\{V_n\}$ for B with each $\bar{V}_n$ not intersects B.

Gn: n, define $U_n' = U_n - \bigcup_{i=1}^{n} \bar{V}_i$ and

$$V_n' = V_n - \bigcup_{i=1}^{n} \bar{U}_i$$

Each set $U_n'$ is open being the difference of an open set $U_n$ and a closed set $\bigcap_{i=1}^{n} \bar{V}_i$.

$|||^{ly}$ each set $V_n$ is open.

Let $U' = \bigcup_{n=1}^{\infty} U_n'$ and $V' = \bigcup_{n=1}^{\infty} V_n'$.

$\because$ $U'$ and $V'$ are open sets.

Claim: $U' \cap V' = \phi$

Suppose $x \in U' \cap V'$

$x \in U$ $\Rightarrow$ $x \in U_j'$ for some j and $x \in V_k'$ for some k.

Suppose that $j \leq k$.

By the def of $V_k'$.

$V_k' = V_k - (\bar{U}_1 \cup \bar{U}_2 \cup \dots \cup \bar{U}_k)$

i.e) $x \in V_k$ & $x \notin \bar{U}_1 \cup \bar{U}_2 \cup \dots \cup \bar{U}_k$

$\Rightarrow x \in V_k$ but $x \notin \bar{U}_j$ for any $i$ ($\because j \le k$)

$\Rightarrow x \notin U_j$ for any $j$

$\Rightarrow x \notin U_j'$ for any $j$.

which is a $\Rightarrow\Leftarrow$ to (1)

A similiar contradiction arises if $j > k$

$\therefore U' \cap V' = \phi$

i.e) $U'$ & $V'$ are open set s.t $A \subset U'$ & $B \subset V'$

and $U' \cap V' = \phi$

$\therefore X$ is normal.

Thm 32.2

Every metrizable space is Normal.

Let $X$ be a metrizable with metric $d$.

Let $A$ and $B$ be a two non-empty disjoint closed subset of $X$.

Let $a \in A \Rightarrow a \notin B$ ($\because A \cap B = \phi$)

$\Rightarrow a \in X - B$ which is open

$\Rightarrow \exists\, \epsilon_a > 0$. s.t $B(a, \epsilon_a)$ $(X - B)$. for each $a \in A$, doesn't intersect $B$.

III$^{ly}$, For each $b \in B$ choose $\epsilon_b > 0$.

We can find an open ball $B(b, \epsilon_b)$ doesn't intersecting $A$.

Let $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$ and

$V = \bigcup_{b \in B} B(b, \epsilon_b/2)$

Clearly U and V are open sets containing A & B respectively.

It is enough to P.T $U \cap V = \phi$

Suppose $U \cap V \neq \phi$

Let $z \in U \cap V$.

i.e. $z \in B\left(a, \frac{\epsilon_a}{2}\right) \cap B\left(b, \frac{\epsilon_b}{2}\right)$ for some $a \in A$ and for some $b \in B$.

i.e. $z \in B(a, \epsilon_a/2)$ for some $a \in A$

$z \in B(b, \epsilon_b/2)$ for some $b \in B$.

$\Rightarrow d(z, a) < \epsilon_a/2$ and $d(z, b) < \epsilon_b/2$

Without the loss of generality we assume $\epsilon_b \leq \epsilon_a$.

By $\Delta^{le}$ inequality.

$d(a,b) \leq (\epsilon_a + \epsilon_b)/2$

$\leq \epsilon_a/2 + \epsilon_b/2$

$< \epsilon_a/2 + \frac{\epsilon_a}{2}$

$\leq \epsilon_a$

$d(a,b) < \epsilon_a$.

i.e. $b \in B(a, \epsilon_a)$

Which is a $\Rightarrow\Leftarrow$ to $b \in B$ and $b \in B(a, \epsilon_a)$

Which doesn't intersects B.

$\therefore U \cap V = \phi$

i.e. $\exists$ disjoint open set U and V. s.t $A \subset U$ & $B \subset$

$\therefore X$ is normal.

Thm 32.3.

Every compact Hausdroff Space is Normal.

Let $X$ be a compact H.S.

$\because X$ is hausdroff, singleton sets are closed.

Let $A, B$ be two disjoint closed subset of $X$.

$\because A$ & $B$ are closed and $X$ is compact.

A and B are compact.

$\because$ closed subset of a compact subspace is compact.

Let $x \in A$

For each $y \in B$ $\exists$ disjoint nbds of $x$ and $y$ respectively

$\{V_y / y \in B\}$ is an open cover for B.

$\because$ B is compact $\exists$ a finite sub-collection.

$\{V_{y_1}, V_{y_2} \dots V_{y_n}\}$ of $\{V_y / y \in B\}$ that covers B.

Let $U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$.

$V = V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$.

$\therefore U$ & $V$ are disjoint open sets. s.t. $A \subset U$ and $B \subset V$.

For each $x \in A$ $\exists$ disjoint open $U_x$ and $V_x$

Containing $x$ & B respectively.

$\{U_x / x \in A\}$ is an open cover for A.

$\because$ A is compact $\exists$ a finite subcollection

$\{U_{x_1}, U_{x_2} \dots U_{x_n}\}$ of $\{U_x / x \in A\}$ that covers A.

Let $U = U_{x_1} \cup U_{x_2} \cup \dots \cup U_{x_n}$

$V = V_{x_1} \cap V_{x_2} \cap \dots \cap V_{x_n}$.

then $U$ & $V$ are disjoint open sets

s.t $A \subset U$ and $B \subset V$ and $U \cap V = \phi$

$\therefore X$ is normal.

Thm 32.4

Every well ordered set is normal in the order topology.

Let $X$ be a well ordered set with order topology. We assume that every interval of the form $(x, y]$ is open in $X$.

$$(x,y] = \begin{cases} [x,y] & \text{if } y \text{ is the Largest element of } X \\ [x,y') & \text{if } y \text{ is the immediate Succesor of } y \text{ in } X. \end{cases}$$

let $A$ and $B$ be a two non-empty disjoint Closed subsets of $X$.

Case (i):

Assume that neither $A$ nor $B$ contains the Smallest element $a_0$ of $X$

(In a well ordered set every non-empty set has smallest element).

Let $a \in A$

$\Rightarrow a \notin B$ $\quad$ $(\because A \cap B = \phi)$

$\Rightarrow a \in X - B$ which is open.

$\Rightarrow \exists$ a basis element $(x, a]$ such that $a \in (x, a] \subset X - B$.

$\Rightarrow (x, a] \cap B = \phi$

For each $a \in A$, $\exists$ a basis element $(x_a, a]$ such that $(x_a, a] \subset X - B$.

i.e) $(x_a, a] \cap B = \phi$ $\forall$ $a \in A$.

$|||^{ly}$ for each $b \in B$ $\exists$ an open set $(y_b, b]$ such that $(y_b, b] \subset X - A$.

i.e., $(y_b, b] \cap A = \phi$.

Let $U = \bigcup_{a \in A} (x_a, a]$

$V = \bigcup_{b \in B} (y_b, b]$

Then clearly $U$ and $V$ are open sets containing $A$ & $B$ resply. (i.e $A \subset U$ & $B \subset V$).

Claim: $U \cap V = \phi$.

Suppose $z \in U \cap V$

Then $z \in (x_a, a]$ for some $a \in A$ and $z \in (y_b, b]$.

If $y_b < a$, then $a \in (y_b, b]$

Which is a $\Rightarrow\Leftarrow$ to $(y_b, b] \cap A = \phi$

$|||^{ly}$ a contradiction occurs if $b < a$.

$\therefore U \cap V = \phi$.

Case (ii):

Let $a$ be the smallest element in $X$ and Let $a_0 \in A$.

$\{a_0\}$ is both open and closed in $X$

($\because$ $X$ with order topology is Hausdroff)

Consider $A_1 = A - \{a_0\}$ then $A_1$ is closed.

$\because$ $\{a_0\}$ is closed.

$\therefore$ $A$ and $B$ are disjoint closed set in $X$.

By case (i) $\exists$ disjoint open sets $U$ and $V$ s.t $A_1 \subset U$ & $B \subset V$.

Let $U_1 = U \cup \{a_0\}$ ($\because a_0$ is open). Then $U_1$ and $V$ are disjoined open sets containing A & B resly.

$\therefore X$ is normal.

Sec : 33 The Urysoln Lemma

Thm: 33.1 (Urysoln Lemma).

Let X be a normal space. Let A and B be disjoint closed subsets of X.

Let $[a,b]$ be a closed interval in the real line then $\exists$ a cts map $f : X \to [a,b]$. s.t $f(x) = a$ for every $x$ in A and $f(x) = b$ for every $x$ in B.

We shall prove the result for $[0,1]$ bcz $[a,b]$ is homomorphic to $[0,1]$.

STEP: 1

Let P be the set of all rational no. in the interval $[0,1]$.

We shall define an open set $U_p$ of $X$ for each $p \in [0,1]$ s.t $\overline{U_p} \subset U_q$ whenever $p<q$.

$\because$ P is countable, we can use induction to define the set $U_p$.

Arrange the elements of P in some order. Let 1 & 0 (zero) be the 1st two element in the arrangeme

Now we define the set $U_p$ as follows 1st define $U_1 = X - B$ which is open such A is closed set contained in the open set $U_1$ and since X is normal $\exists$ an open set $U_0$ such that

$$A \subset U_0 \subset \overline{U_0} \subset U_1$$

In general, Let $P_n$ denote the set consisting of the $1^{st}$ n. rational number in the arrangements.

Suppose that $U_p$ is defined for all rational no. P, belonging to the set in satisfying the condition

$$P<q \Rightarrow \bar{U_p} \subset \bar{U_q} \longrightarrow (1)$$

Let r denote the next rational number the arrangement we wish to define $U_r$.

Consider the set $P_{n+1} = P_n \cup \{r\}$ then $P_{n+1}$ is a finite subset of the interval $[0,1]$.

Which is a simply ordered set.

$\therefore$ $P_{n+1}$ is also an simply ordered.

In a finite ordered set every element other than the smallest and Largest has an immediate predecessor and an immediate successor.

The no. '0' is the smallest element and 1 is the Largest element of the simply ordered set $P_{n+1}$ but r is neither 0 nor 1.

$\therefore$ r has an immediate predessor P and an immediate successor q in $P_{n+1}$.

i.e. $P<r<q$.

The sets $U_p$ & $U_q$ are already defined on $\bar{U_p} \subset U_2$

( by the induction hypothesis).

$\because$ X is normal.

$\exists$ an open set $U_r$ of X, s.t.

$$\bar{U_p} \subset U_r \subset \bar{U_r} \subset U_2.$$

Next we assert that eqn (1) holds for every pair of element of $P_{n+1}$.

i) If both element lie in the (1) holds by the induction hypothesis.

ii) If one of them is $r$ & the other point is $s$ of $P_n$. Then either $s < p$.

In which case $\overline{U_s} \subset U_p \subset \overline{U_p} \subset U_r$.

$\therefore \overline{U_s} \in U_r$.

(or) $s \geq q$ In which case $\overline{U_r} \subset U_q \subset \overline{U_q} \subset U_s$

$\therefore \overline{U_r} \subset U_s$.

Thus for every pair of element of $P_{n+1}$ eqn (1) holds by induction we have defined $U_p$ for every $p \in P$.

STEP 2:

Now we shall defined open set $U_p$ to all rational no. $p$ in $\mathbb{R}$ by defining $U_p = \phi$ if $p < 0$ and $U_p = X$ if $p > 1$.

$U_p = \phi$ if $p<0$
$U_p = X$ if $p>1$

Claim: If $p$ and $q$ are any two rational numbers s.t $p < q$ then $\overline{U_p} \subset U_q$.

Case (i)

If $p$ and $q$ are $< 0$ then

$U_p = U_q = \phi$

$\therefore \overline{U_p} = \phi = \phi \subset \phi = U_q$

i.e) $\overline{U_p} \subset U_q$.

case (ii):

If $p<0$ and $q \in [0,1]$ then $U_p = \phi$

$\therefore \overline{U_p} = \bar{\phi} = \phi \subset U_q$.

i.e) $\overline{U_p} \subset U_q$.

Case (iii)

$p$ & $q \in [0,1]$. then by step 1 $\overline{U_p} \subset U_q$.

Case iv):

If $p \in [0,1]$ & $q>1$ then $U_q = X$ and

$\overline{U_p} \in X = U_q$

$\therefore U_p \subset U_q$

Case v):

If $p$ and $q>1$ then $U_p = U_q = X$

$\overline{U_p} = \bar{X} = X \subset X = U_q$

$\therefore \overline{U_p} \subset U_q$.

From the above 5 cases it is true that for any pair of rational no. $p$ & $q$

$p<q \Rightarrow \overline{U_p} \subset U_q$.

STEP: 3.

Gn a pt $x \in X$ define $Q(x) = \{p \in Q / x \in U_p\}$ be the set of rational number.

i.e) $x \in U_p \Rightarrow p \in Q$

For every $p>1$, $x \in X = U_p$

$\therefore$ Every $p>1 \in Q(x)$

For every $p<0$, $x \notin \phi = U_p$

$\therefore$ Every $p<0 \notin Q(x)$

$\therefore Q(x)$ is bounded below and its greatest lower bound (glb) is a pt of $[0,1]$.

Define $F(x) = \inf Q(x)$ then $f$ is a fun from $f: X \to [0,1]$.

STEP 4:

We shall show that $f$ is the desired fun.

Claim:

$f(x) = 0 \ \forall \ x \in A$ and $f(x) = 1 \ \forall \ x \in B$

Let $x \in A$

$\therefore x \in U_p \ \forall \ P \geq 0$

$\therefore P \in Q(x) \ \forall \ P \geq 0$

$\therefore$ glb of $Q(x) = 0$

i.e) $f(x) = 0 \quad \forall \ x \in A$

Let $x \in B$ then $x \notin U_p \ \forall \ P \leq 1$.

$\therefore P \notin Q(x) \ \forall \ P \leq 1$

$\therefore$ glb of $Q(x) = 1$

i.e) $f(x) = 1 \ \forall \ x \in B$.

Hence the claim.

STEP 5:

Next we shall s.t $S: X \to [0,1]$ is cts for this Purpose we first prove the following result.

i) $x \in \overline{U_r} \Rightarrow f(x) \leq r$

ii) $x \notin U_r \Rightarrow f(x) \geq r$.

$x \in \overline{U_r} \Rightarrow X \in U_s \ \forall \ S > r$;

$\Rightarrow S \in Q(x) \ \forall \ S > r$

$\Rightarrow$ glb of $Q(x) \leq r$.

$\Rightarrow f(x) \le r$

Hence result (i)

$x \notin U_r \Rightarrow x \notin U_s \ \forall \ s < r$

$\Rightarrow s \notin Q(x) \ \forall \ s < r$

$\Rightarrow$ glb of $Q(x) \ge r$

$\Rightarrow f(x) \ge r$

Hence result (ii)

Finally we prove $f: X \to [0,1]$ is cts.

Let $x_0 \in X$

let B be any nbd of $f(x_0)$ in $[0,1]$.

$\therefore \exists$ a basis element $(c,d)$ s.t $f(x_0) \in (c,d) \subset V$

$\therefore c < f(x_0) < d$.

Choose 2 rational pts $p$ & $q$. s.t

$c < p < f(x_0) < q < d$.

Let $U = U_q - \overline{U_p}$

$\therefore$ U is open.

Claim:

$x_0 \in U$

For if $x_0 \notin U$ then $x_0 \notin U_q - \overline{U_p}$

$\therefore x_0 \notin U_q$ (or) $x_0 \in \overline{U_p}$

$x_0 \notin U_q \Rightarrow f(x_0) \ge q$ (by ii)

Which is a $\Rightarrow\Leftarrow$ to $f(x_0) < q$.

$x_0 \in \overline{U_p} \Rightarrow f(x_0) \le p$ (by (i))

which is a $\Rightarrow\Leftarrow$ to the fact that $f(x_0) > p$

$\therefore x_0 \in U$

i.e., U is a nbd of $x_0$

Claim: $f(U) \subset V$

Let $f(x) \in f(U)$

$\therefore x \in U$

i.e., $x \in U_q - \overline{U_p}$

$\Rightarrow x \in U_q$ and $x \notin \overline{U_p}$

$x \in U_q \Rightarrow x \in \overline{U_q}$

$\Rightarrow f(x) \le q$ (by (i))

$x \notin \overline{U_p} \Rightarrow x \notin U_p$

$\Rightarrow f(x) \ge p$ (by (ii))

$\therefore p \le f(x) < q$

$\Rightarrow f(x) \in [p, q]$

$\Rightarrow f(x) \in [c, d]$

$\therefore f(x) \in V$

$\therefore f(x) \in f(U) \Rightarrow f(x) \in V$.

$\therefore f(U) \subset V$

$\because x_0$ is arbitrary

$f: X \to [0,1]$ is cts.

Hence: Urysohn lemma. CIA - III Portion.

Def:

If A and B are two subsets of the topological space X if there is a cts fun $f: X \to [0,1]$ s.t. $f(A) = \{0\}$, $f(B) = \{1\}$. We say that A & B can be separated by a cts fun.

Def: Completely Regular

A space X is Completely regular if one pt sets are closed in X and if for each pt $x_0$ and each Closed set A not containing $x_0$ there is a cts fun

$f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$

Results (or) Eg:

1) Any normal space is completely regular.

Let $X$ be a normal space.

T.P: $X$ is completely regular

Let $a \in X$.

Let $B$ be any closed set in $X$ such that $a \notin B$.

In a normal space one pts set are closed.

$\therefore \{a\}$ is closed.

$\because a \notin B$, $\{a\}$ & $B$ are disjoined closed sets in $X$.

By applying Urysohn lemma,

$\exists$ a cts fun $f: X \to [0,1]$ such that

$f(\{a\}) = \{1\}$ & $f(B) = \{0\}$

i.e) $f(a) = 1$ and $f(B) = 0$

$\therefore X$ is completely regular.

2) Any completely regular space is regular,

Let $X$ be a completely regular space.

T.P: $X$ is regular.

Let $a \in X$ & $B$ be any closed set in $X$. Such that $a \notin B$.

$\because X$ is completely regular, $\exists$ a cts fun $f: X \to [0,1]$.

S.t $f(a) = 1$ and $f(B) = \{0\}$

Let $U = f^{-1}[[0, 1/2)]$ and $V = f^{-1}[(1/2, 1]]$

$\because [0, 1/2)$ & $(1/2, 1]$ are open in $[0,1]$ in $X$.

i.e) $U$ & $V$ are open in $X$.

Also, U & V are disjoined bcz $[0, \frac{1}{2})$ & $(\frac{1}{2}, 1]$ are disjoint.

$\therefore$ U & V are the required disjoint open sets.

$\therefore$ X is regular.

$\therefore$ Any completely regular space is space.

Thm 33.2

i) A subspace of a completely regular space is completely regular and

ii) Product of completely regular space is completely regular.

Proof:

i) Let $X$ be a completely regular space.

Let $Y$ be a subspace of $X$.

T.P: $Y$ is completely regular.

One pt set in $Y$ are closed in $Y$,

for let $y \in Y$ then closure of $\{y\}$ in $Y = \overline{\{y\}} \cap Y$

$= \{y\} \cap X$

(∵ One pt sets are closed in $X$.

∴ $\overline{\{y\}} = \{y\}$)

∴ $\{y\}$ is closed in $Y$.

Let $y_0 \in Y$ and Let $B$ be a closed set in $Y$ s.t $y_0 \notin B$.

$B = C \cap Y$ where $C$ is closed in $X$

$y_0 \notin B \Rightarrow y_0 \notin C$

∵ $X$ is completely regular.

∃ a cts function $f: X \to [0,1]$ such that $f(B) = \{0\}$ and $f(y_0) = 1$.

Now $f/y$ is the required cts fun from $Y$ to $[0,1]$ s.t

$(f/y)B = \{0\}$ and $(f/y) y_0 = 1$

∴ $Y$ is completely regular.

iii). Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of completely regular spaces.

T.P: $\prod_{\alpha \in J} X_\alpha$ is completely regular.

$\because$ each $X_\alpha$ is completely regular.

i.e) $X_\alpha$ is regular.

$\therefore$ One pt set in $\prod X_\alpha$ is closed.

Let $b = (b_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$.

Let $B$ be any closed set in $\prod X_\alpha$. s.t $b \notin B$

Let $U = X - B$ then $U$ is an open set containing the point $b$.

$\therefore \exists$ a basis element $\prod_{\alpha \in J} U_\alpha$. s.t $b \in \prod U_\alpha \subset U$.

Where, $U_\alpha = U_{\alpha_i}$ for $\alpha = \alpha_1, \alpha_2 .. \alpha_n$

$= X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2 .. \alpha_n$ for $i = 1, 2 .. n$,

$b_{\alpha_i} \in U_{\alpha_i}$.

$\therefore b_{\alpha_i} \notin X_{\alpha_i} - U_{\alpha_i}$ which is

$\because X_{\alpha_i}$ is completely regular $\exists$ a cts fun.

$f_i : X_{\alpha_i} \to [0,1]$ s.t $f_i : (X - U_\alpha) = \{0\}$ and

$f_i(b_{\alpha_i}) = 1$

Let $\phi_i(x) = f_i(\pi_{\alpha_i}(x))$ if $x \in B$ then

$x_{\alpha_i} \in X_{\alpha_i} - U_{\alpha_i} = f_i(x_{\alpha_i})$

$= 0$

then $\phi_i$ maps $\prod X_\alpha$ continuously into $\mathbb{R}$.

Def $\phi : \prod X_\alpha \to [0,1]$. s.t

$\phi(x) = \phi_1(x) \times \phi_2(x) \times \ldots$

then $\phi$ is cts.

If $x \in B$ then $\phi(x) = \phi_1(x) \times \phi_2(x) \times \cdots \times \phi_n(x)$

$= 0$

If $x = b$ then $\phi(x) = \phi_1(b) \times \phi_2(b) \times \cdots \times \phi_n(b)$

$= [1] \times \cdots [1] = 1$

$\therefore$ $\phi$ is the required cts function

$\therefore \prod_{\alpha \in J} X_\alpha$ is completely regular

Sec 34 The Urysohn Metrization Thm

Thm 34.1 [UMT]

Every regular space $X$ with a countable basis $X$ is

reg metrizable.

Proof:

STEP 1:

We prove the following $\exists$ a countable collection of cts fun $f_n : X \to [0,1]$ having the property that given any pt $x_0$ of $X$ and any nbd $U$ of $x_0$ $\exists$ an index $n$ such that $f_n$ is +ve at $x_0$ and vanish outside $U$.

Let $B = \{B_n / n \in \mathbb{Z}_+\}$

$\because$ $X$ is regular and has countable basis

$X$ is normal.

$\therefore$ $\forall$ $B_n \in B$ ; $\exists$ $B_m \in \mathcal{B}$, s.t $\overline{B_n} \subset B_m$

$\therefore$ $\overline{B_n}$ and $X - B_m$ are disjoint closed set in $X$

$\because$ $X$ is normal by Urysohn Lemma.

$\exists$ a cts fun $g_{n,m} : X \to [0,1]$

Such that:

$g_{n,m}(x) = 1$ if $x \in \overline{B_n}$

$= 0$ if $x \in X - B_m$

Let $x_0 \in X$ and let $U$ be a nbd of $x_0$

$\because$ $\mathcal{B}$ is a basis for $X$, $\exists$ a basis element $B_m \in \mathcal{B}$ s.t $x_0 \in B_m \subset U$.

$\because$ $X$ is normal, $\exists$ a basis element $B_n$ such that

$x_0 \in B_n \subset \bar{B}_n \subset B_m \subset U$.

$\therefore$ $\exists$ a cts fun $g_{n,m} : X \to [0,1]$. s.t

$g_{n,m}(x) = 1$ if $x \in \bar{B}_n$

$= 0$ if $x \in X - B_m$.

$\therefore$ $g_{n,m}(x_0) = 1 > 0$ $(\because x_0 \in B_n \subset \bar{B}_n)$

i.e) $g_{n,m}(x_0) > 0$

i.e) $g_{n,m}$ is +ve at $x_0$

If $x \notin U$ then $x \in X - U$

$x \in X - B_m$

$\therefore$ $g_{n,m}(x) = 0$.

i.e) $g_{n,m}$ vanishes outside $U$.

Let $g_{n,m} = f_n$

The Collection $\{f_n\}_{n \in \mathbb{Z}_+}$ is the required Countable collection of cts fun the above pts.

STEP 2 :

From step 1 we get a countable collection $\{f_n\}$ of cts fn satisfy the above pts.

Consider the $\mathbb{R}^{\omega}$ in the product topology.

Def the map $F : X \to \mathbb{R}^{\omega}$. by

$F(x) = (f_1(x), f_2(x) \dots f_n(x) \dots)$

We assert that $F$ is an imbedding.

$F$ is cts because each $f_n$ is countable and $\mathbb{R}^{\omega}$ has the product topology.

$\therefore$ $F$ is 1-1

let $x \neq y \in X$

$\because X$ is regular, $\{y\}$ is closed.

Let $U = X - \{y\}$ then $x \in U$.

By step 1, $\exists$ a cts function $f_N$ s.t $f_N(x) > 0$ and $f_N(y) = 0$

$f_N(x) \neq f_N(y)$

$\therefore F(x) \neq F(y)$

$\therefore x \neq y \Rightarrow F[x] \neq F[y]$

$\therefore F$ is 1-1

$F: X \to F(X)$ is onto.

$\therefore F$ is bijection.

Next we shall prove $F$ is an open map.

Let $U$ be an open in $X$.

T.P : $F(U)$ is open in $F(X)$.

Let $F(X) = Z$

Let $z_0 \in F(U)$ then $z_0 = F(x_0)$ for some $x_0 \in U$.

$\therefore \exists$ a cts fun $f_N : X \to [0,1]$ such that

$f_N(x_0) > 0$ and $f_N(x) = 0 \ \forall \ x \in X - U$

Let $V = \pi_N^{-1}(0, \infty)$ then $V$ is an open subset of $\mathbb{R}$

Let $W = V \cap Z$ then $W$ is open in $F(X)$.

CLAIM:

$z_0 \in W$ and $W \subset F(U)$

$x_0 \in U$ and $f_N(x_0) > 0$

$f_N(x_0) \in (0, \infty)$

$\pi_N^{-1}(f_N(x_0)) \in \pi_N^{-1}(0, \infty)$ ($\because \pi_N \circ F = f_N$)

i.e) $F(x_0) \in \pi_N^{-1}(0, \infty) = V$

$\Rightarrow F(x_0) \in V$

i.e. $z_0 \in V$

Also $z_0 \in F(X) = Z$

$\therefore z_0 \in V \cap Z$

i.e. $z_0 \in W$

Let $y \in W$

$\therefore y \in V \cap Z = V \cap F(X)$

$\therefore y \in V$ and $y \in F(X)$

$y \in V \Rightarrow y \in \pi_N^{-1}(0, \infty)$.

$y \in F(x) \Rightarrow y = F(x)$ where $x \in X$.

$y \in \pi_N^{-1}(0, \infty) \Rightarrow F(x) \in \pi_N^{-1}(0, \infty)$

$\Rightarrow \pi_N \circ F(x) \in (0, \infty)$

$\Rightarrow f_N(x) \in (0, \infty)$

$\Rightarrow f_N(x) > 0$

$\therefore x \notin X - U$

$\therefore x \in U$

$\therefore F(x) \in F(U)$

$\Rightarrow y \in F(U)$

$\therefore y \in W \Rightarrow y \in F(U)$

$\therefore W \subset F(U)$

$\therefore F(U)$ is open

$\therefore F$ is an open map.

$\therefore F$ is an imbedding of $X$ in $\mathbb{R}^\omega$ but

$\mathbb{R}^\omega$ is metrizable

$\therefore F$ is metrizable.

Thm : 34.2 (Imbedding thm).

Let $X$ be a space in which one-point sets are closed. Suppose that $(f_\alpha)_{\alpha \in J}$ is an indexed Family of cts funs $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each pt $x_0$ of $X$ and each ngd $U$ of $x_0$. There is an index $\alpha$ such that $f_\alpha$ is +ve at $x_0$ and vanishes outside $U$, then the function $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))$ is an imbedding of $X$ in $\mathbb{R}^J$. If $f_\alpha$ maps for each $\alpha$ then $F$ imbeds $X$ in $[0,1]^J$.

Proof.

A countable collection.

Consider $\mathbb{R}^J$ with the product topology.

Def a map $F : X \to \mathbb{R}^J$ by $F(x) = (f_\alpha(x))_{\alpha \in J}$

We assert that, $F$ is an imbedding.

$F$ is cts because each $f_\alpha$ is cts and $\mathbb{R}^J$ is with the product topology, $F$ is 1-1

For let $x \neq y \in X$

$\because X$ is regular, $\{y\}$ is closed.

Let $U = X - \{y\}$ then $x \in U$.

By hypothesis, $\exists$ a cts then $F_\alpha$ s.t $f_\alpha(x) > 1$ and $f_\alpha(y) = 0$.

$$f_\alpha(x) \neq f_\alpha(y)$$

$$\therefore F(x) \neq F(y)$$

$$\therefore x \neq y \Rightarrow F(x) \neq F(y)$$

$\therefore F$ is 1-1

$F : X \to F(X)$ is onto.

$\therefore F$ is a bijection.

Next we shall prove that $F$ is an open map

Let $U$ be open in $X$

T.P: $F(U)$ is open in $F(X)$

Let $F(X) = Z$

Let $z_0 \in F(U)$ then $z_0 = F(x_0)$ for some $x_0 \in U$

$\therefore \exists$ a cts fun $f_\alpha : X \to [0,1]$ s.t

$f_\alpha(x_0) > 1$ & $f_\alpha(x) = 0 \ \forall x \in X - U$.

Let $V = \pi_\alpha^{-1}(0,\infty)$ then $V$ is an open subset of $\mathbb{R}^J$.

Let $W = V \cap Z$. then $W$ is open is $F(X)$.

Claim: $z_0 \in W \subset F(U)$

$x_0 \in U \Rightarrow \therefore f_\alpha(x_0) > 0$

$f_\alpha(x_0) \in (0,\infty)$

$\pi_\alpha^{-1}(f_\alpha(x_0)) \in \pi_\alpha^{-1}(0,\infty)$

i.e.) $F(x_0) \in \pi_\alpha^{-1}(0,\infty) = V$

$\Rightarrow F(x_0) \in V$

$\Rightarrow z_0 \in V$

Also $z_0 \in F(X) = Z$

$z_0 \in V \cap Z$

$\Rightarrow z_0 \in W$

Let $y \in W$.

$y \in V \cap Z \Rightarrow V \cap F(X)$

$y \in V \Rightarrow y \in \pi_\alpha^{-1}(0,\infty)$

$y \in F(X) \Rightarrow y = F(x)$ where $x \in X$.

$y \in \pi_\alpha^{-1}(0,\infty) \Rightarrow F(x) \in \pi_\alpha^{-1}(0,\infty)$

$\Rightarrow (\pi_\alpha \circ F)(x) \in (0,\infty)$

$\Rightarrow f_\alpha(x) \in (0,\infty)$

$f_\alpha(x) > 0$

i.e> $x \notin X - U$

$\therefore x \in U$

$\therefore F(x) \in F(U)$

i.e>. $y \in F(U)$

$F \in U \Rightarrow y \in F(U)$

$\therefore W \subset F(U)$

$\therefore F(U)$ is open

$\therefore F$ is an open map

$\therefore F$ is an imbedding of $X$ into $\mathbb{R}^J$.

Thm 34.3

A Space $X$ is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some $J$

The Tietze Extension Thm:-

SSt:

Let $X$ be a normal space. Let $A$ be a closed subspace of $X$.

a> Any cts map of $A$ into the closed interval $[a,b]$ of $\mathbb{R}$ may be extended to a cts map of all of $X$ into $[a,b]$.

b> Any cts map of $A$ into $\mathbb{R}$ may be extended to the cts map of all of $X$ into $\mathbb{R}$.

Proof:

STEP: 1

Let $f: A \to [-r, r]$ be a cts fun.

We assert that $\exists$ a cts fun $g: X \to \mathbb{R}$ s.t $|g(x)| \leq r/3 \ \forall \ x \in X$ and $|-f(a) + g(a)| \leq 2r/3 \ \forall \ a \in A$.

Divide the interval $[-r, r]$ into 3 equal subinterval of length $2r/3$.

$I_1 = [-r, -r/3]$

$I_2 = [-r/3, r/3]$

$I_3 = [r/3, r]$

Let $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$

$\because$ $I_1$ and $I_3$ are disjoint closed interval and

$\because$ $f : A \to [-r, r]$ is cts.

$f^{-1}[I_1]$ and $f^{-1}[I_3]$ are disjoint closed sets

in A.

i.e. B and C are disjoint closed sets in

A but A is closed in X.

$\therefore$ B and C are closed in X which are disjoint

By the Urysohn Lemma,

$\exists$ a cts fun $g : X \to [-r/3, r/3]$ s.t

$g(B) = \{\frac{-r}{3}\}$ and $g(C) = \{\frac{r}{3}\}$

$\therefore g(x) \in \{\frac{-r}{3}, \frac{r}{3}\}$

$\therefore g(x) \in \{\frac{-r}{3}, \frac{r}{3}\}$

$\therefore |g(x)| \leq \frac{r}{3} \ \forall \ x \in X \longrightarrow (1)$

Next we assert that $|f(a) - g(a)| \leq \frac{2r}{3} \ \forall \ a \in A$.

There are 3 cases.

Case (i):

If $a \in B$ then $f(a)$ and $g(a) \in I_1$

$|f(a) - g(a)| \leq$ the width of $I_1 = \frac{2r}{3}$

$\therefore |f(a) - g(a)| \leq \frac{2r}{3}$.

case (ii)

If $a \in C$ then $f(a)$ and $g(a) \in I_3$

$|f(a) - g(a)| \leq$ width of $I_3 = \frac{2r}{3}$

$\therefore |f(a) - g(a)| \leq \frac{2r}{3}$.

Case (iii)

If $a \notin B \cup C$ then $f(a)$ and $g(a) \in I_2$

$\therefore |f(a) - g(a)| \leq$ width of $I_2 = \frac{2r}{3}$

$\therefore |f(a) - g(a)| \leq \frac{2r}{3}$

$\therefore |f(a) - g(a)| \leq \frac{2r}{3} \;\forall\; a \in A$.

Hence step 1 (from 1 & 2)

STEP 2 :

Next we prove result (a) of Tietz extension thm, without the Loss of generality we can replace the arbitrary closed interval $[a, b]$ the Closed interval $[-1, 1]$ because any 2 closed intervals are homeomorphic.

Let $f : A \to [-1, 1]$ be a cts function.

Then by step 1

$\exists$ a cts func. $g_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ s.t

$|g_1(x)| \leq \frac{1}{3} \;\forall\; x \in X$

$|f(a) - g_1(a)| \leq \frac{2}{3} \;\forall\; a \in A$.

Now Consider the function.

$f - g_1 : A \to [-\frac{2}{3}, \frac{2}{3}]$

Applying step 1 again & Letting $r = \frac{2}{3}$.

We obtain a cts fn $g_2 : X \to [-\frac{1}{3}\frac{2}{3}, \frac{1}{3}\frac{2}{3}]$

Such that $|g_2(x)| \leq \frac{1}{3}\frac{2}{3}$ $\forall x \in X$ and

$|f(x) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2$ $\forall a \in A$.

Then we apply step ① to the fun $f - g_1 - g_2$.

And so on.

At the general step we have real-value fun $g_1, g_2 \cdots g_n$ defined on all of X such that

$|f(a) - g_1(a) \cdots g_n(a)| \leq \left(\frac{2}{3}\right)^n$ $\forall a \in A$.

Applying step ① to the function $f - g_1 \cdots - g_n$ with $r = \left(\frac{2}{3}\right)^n$ we obtain a real value function $g_{n+1}$ defined on all X such that.

$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ for $x \in X$

$|f(a) - g_1(a) \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1}$ for $a \in A$.

By induction, the fun $g_n$ are defined for all n

Now we define,

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

For all x in X of course we have to know this infinite series Cgs,

$$\Rightarrow \frac{1}{3}\left(\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}\right)$$

To show that g is cts, we must the sequence $s_n$ Cgs to g uniformly.

If $k > n$ then

$$|S_k(x) - S_n(x)| = \left|\sum_{i=n+1}^{k} g_i(x)\right|$$

$$\le \frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1}$$

$$\le \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1}$$

$$\le \left(\frac{2}{3}\right)^{n}$$

Holding $n$ fixed and letting $k \to \infty$ we see that

$$|g(x) - S_n(x)| \le \left(\frac{2}{3}\right)^{n} \quad \forall\, x \in X$$

$\therefore$ $S_n$ cgs to $g$ uniformly.

We show that $g(a) = f(a)$ for $a \in A$.

Let $S_n(x) = \sum_{i=1}^{n} g_i(x)$ the $n^{th}$ partial sum of the series. Then $g(x)$ is by definition the limit of the infinite seq $S_n(x)$ of partial sums

$\therefore$ $\left|f(a) - \sum_{i=1}^{n} g_i(a)\right| = |f(a) - S_n(a)| \le \left(\frac{2}{3}\right)^{n}$

$\forall$ a in A. It follows that

$$S_n(a) \to f(a) \;\forall\, a \in A.$$

$\therefore$ We have $f(a) = g(a)$ for $a \in A$.

Finally we show that $g$ maps $X$ into the interval $[-1, 1]$. This condition is infact satisfied automatically since the series $\frac{1}{3}\sum\left(\frac{2}{3}\right)^{n}$ cgs to 1.

STEP : 3 :

We now prove part (b) of the thm in which $f$ maps $A$ into $\mathbb{R}$. We can replace $\mathbb{R}$ by the open interval $(-1, 1)$.

$\because$ this interval is homeomorphic to $\mathbb{R}$

So let $f$ be a cts map from $A$ into $(-1, 1)$.

The half of the Tietz thm already proved show that we can extended $f$ to a cts map $g : X \to [-1, 1]$ mapping $X$ into the closed interval.

Given $g$, Let us define a subset $D$ of $X$ by the eqn.

$$D = g^{-1}(\{-1\} \cup g^{-1}\{1\})$$

$\because$ $g$ is cts, $D$ is a closed subset of $X$.

Because $g(A) = f(A)$ which is contained in $(-1, 1)$, the set $A$ is disjoint from $D$.

By the Urysoln lemma,

there is a cts fn $\phi : X \to [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. Define

$$h(x) = \phi(x)\, g(x)$$

Then $h$ is cts, being the product of two cts functions. Also, $h$ is an extension of $f$,

$\because$ for $a$ in $A$.

$$h(a) = \phi(a)\, g(a) = 1, \quad g(a) = f(a).$$

Finally, $h$ maps all of $X$ into the open interval $(-1, 1)$. For if $x \in D$, then $h(x) = 0$, $g(x) = 0$.

And, if $x \notin D$, then $|g(x)| < 1$.

It follows that $|h(x)| \leq 1$, $|g(x)| \leq 1$.

**Additional Resource :**

http://mathforum.org

http://ocw.mit.edu/ocwweb/Mathematics

http://www.opensource.org

http://en.wikipedia.org

**Practice Questions:**

**Question Bank**

**Section – A**

1.Define Countability axiom

2.Define First countablity axion

3.Define Second countability axioms

4.Define Regular.

5.Define normal Space.

6.State Urysohn Lemma.

7.Define completely regular.

8.State Imbedding theorem

9.State urysohn metrization theorem

10.State Tietze extension theorem.

**Section – B**

1.P.T a subspace of a regular space is regular ; a product of regular spaces is regular.

2.Let X be a topological space.Let one-point sets in X be closed .Prove that X is regular if and only if given a point x of X and a neighbourhood $U\ of\ x$ ,there is A neighbourhood $V\ of\ x\ such\ that\ \bar{V} \subset U$

3.Prove that a subspace of a completely regular space is completely regular.

4.Prove that every compact Hausdorff space is normal

5.Prove that a subspace of a Hausdorff space is Hausdorff and a product of Hausdorff space is Hausdorff.

6.State the second countability axiom. Prove that it is well behaved with respect to the operations of taking subspaces or countable product.

7.Prove that every metrizable space is normal.

8.Show that a closed subspace of normal space is normal.

9.Suppose that X has a countable basis.Then;

(a) every open covering of X contains a countable subcollection covering X

(b) There exists a countable subset of X that is dense in X

10.Define Lindelof space and prove that the product of two Lindelof spaces need not be Lindelof.

11.P.T a subspace of a Lindelof space need not be Lindelof

12.P.T if X is normal if and only if given a closed set A and an open set $U$

Containing A, there is an open set $V$ containing A such that $\bar{V} \subset U$

13. Prove that the space $\mathbb{R}_k$ is Hausdorff but not regular.

14.P.T every well-ordered set X is normal in the order topology.

**Section – C**

1.P.T every regular space with a countable basis is normal

2.State and prove Urysohn lemma.

3.State and Prove Urysohn metrization theorem

4.State and Prove Imdedding theorem.

5.State and prove Tietze extension theorem.

6.Prove that the following

(a) A subspace of a first countable space is first countable

(b) Countable product of first –countable spaces is first countable.

(c) A Subspace of a second countable space is second countable and a product of second countable spaces is scond countable.

7.Let X be a topological space,

(a) Let A be a subset of X.If there is a sequence of points of A converging to x, Then $x \in \bar{A}$ ,The converse holds if X is first countable.

(b) Let $f; X \to Y$ if $f$ is continuous,then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.The converse holds if X is first countable.

**Recommended Text :** James R. Munkres, Topology (2nd Edition) Pearson Education Pve. Ltd., Delhi-2002 (Third Indian Reprint)