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II M.Sc Mathematics – Semester - III

E-Notes (Study Material)

Core Course : Topology **Code: 23PMA33**
UNIT-III: Connectedness: Connected spaces- connected subspaces of the Real line – Components and local connectedness.
Learning Objectives: To study Connectedness: Connected spaces- connected subspaces of the Real line – Components and local connectedness.
Course Outcome: Understanding connected spaces and their properties. Learning about connected subspaces of the real line. Identifying components (largest connected subsets) in a space. Exploring local connectedness and its significance.

Overview:

Connected Spaces – A space is connected if it cannot be split into two disjoint, non-empty open sets.

Connected Subspaces of the Real Line – Intervals in the real number line (like [a, b]) are always connected.

Components – The largest connected subsets of a space are called components.

Local Connectedness – A space is locally connected if small neighborhoods around each point are connected.

These concepts help in understanding the continuity and structure of spaces in topology.

UNIT - III.

CONNECTEDNESS

Connected spaces :

Let X be a topological space. A separation of X is a pair (U, V) of disjoint non-empty open subsets of X whose union X.

The space X is said to be connected if there doesn't exist a separation of X.

Note :

i) Connectedness is a topological property.

ii) A space X is connected iff the only subset of X that are both open and closed in X are the empty set in X itself.

Proof :

Let X be a connected space.

T.P : ϕ and X are only subset of X which are both open and closed.

If A is non-empty proper set of X which is both open and closed in X, then the set $U = A$ and $V = X - A$

$\because$ A is closed.

$X - A$ is open.

$X = A \cup (X - A)$

Where A and $X - A$ are disjoint non-empty open subset of X.

$\therefore$ X has a separation.

$\Rightarrow$ X is not connected.

$\therefore$ ϕ and X are the only subset of X which are both open and closed.

Conversely,

Let ϕ and X be the only subset of X which are both and closed.

Suppose X is not connected than X has a separation.

$\therefore X = U \cup B$

Where $U \neq \phi$, $B \neq \phi$, $U \cap B = \phi$

U and B are open in X.

$U = (x - b)$ which is closed.

U is both open and closed.

which is $\Rightarrow\Leftarrow$ to hypothesis.

$\therefore$ X is connected space.

Lemma 23.1

If Y is a subspace of X, a separation of Y is a dis. pair of disjoint non empty subset A & B whose union is Y, neither of which contains a limit point of other the space Y. The space Y is connected if $\nexists$ no separation of Y.

Proof:

Suppose that A and B form a separation of Y

then $Y = A \cup B$.

where $A \neq \phi$, $B \neq \phi$. $A \cap B = \phi$.
and A and B are open in Y.
$A = Y - B$ is closed in Y.
(B is open).
If $\bar{A}_y$ is a closure of A in Y.
Then $\bar{A}_Y = A$.
$\therefore$ A is closed in Y.
Also $\bar{A}_y = \bar{A} \cap Y$
Where $\bar{A}$ is the closure of A in X
$\because$ $\bar{A}$ is closed in Y.
$A = \bar{A} \cap Y$
$A = \bar{A} \cap (A \cup B)$
$\Rightarrow (\bar{A} \cap A) \cup (\bar{A} \cap B)$
$A \cup (\bar{A} \cap B)$
$\therefore \bar{A} \cap B = \phi \longrightarrow ①$
But $\bar{A} = A \cup A'$
where A' is the limit point of A.
$\Rightarrow (A \cup A') \cap B = \phi$
$\Rightarrow (A \cap B) \cup (A' \cap B) = \phi$
$\Rightarrow \phi \cup (A' \cap B) = \phi$
$\Rightarrow A' \cap B = \phi$
$\therefore$ ~~we~~ B contains no limit points of A

III ^{ly}

A doesn't contain the limit point of B.

$\therefore$ Y is connected

Conversely,

Suppose that $Y = A \cup B$

where $A \neq \phi$ & $B \neq \phi$

$A \cap B = \phi$, A and B are not containing the limit point of others.

T.P: A and B are open in Y.

By hypothesis, $A' \cap B = \phi$ and $A \cap B' = \phi$

$\Rightarrow \bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$

This both A and B are closed in Y.

Also $A = Y - B$

$\Rightarrow$ A is open in Y.

$B = Y - A$

$\Rightarrow$ B is open in Y.

$\Rightarrow$ Y has a separation.

$\Rightarrow$ Y is not connected.

Hence the subspace of Y is connected if Y has no separation.

OQ 5M Lemma 23.2

If the set C & D is form a separation of X & if Y is connected subspaces of X then Y lies entirely btw either C or D.

Proof:

Gn: C & D from a separation of X.

$\therefore X = C \cup D$ where $C \neq \phi, D \neq \phi$. $C \cap D = \phi$

$\because$ C and D are open in X.

$\Rightarrow C \cap Y$ and $D \cap Y$ are open in Y.

Also, $Y = (C \cap Y) \cup (D \cap Y)$

where $(C \cap Y) \cap (D \cap Y) = \phi$

If $C \cap Y \neq \phi$ and $D \cap Y \neq \phi$ then $C \cap Y$ & $D \cap Y$ will form a separation for Y.

$\therefore$ Y cann't be connected.

Which is $\Rightarrow \Leftarrow$ to Y is connected.

$\therefore$ We must have $C \cap Y = \phi$ (or) $D \cap Y = \phi$.

i.e., $Y \subset D \cap Y$ (or) $Y \subset C \cap Y$.

$\Rightarrow Y \subset D$ (or) $Y \subset C$.

Hence, Y lies entirely btw either C or D.

5M 12Q R. Thm 23.3

The union of a collection of connected subspace of X that have a point in common is connected.

Proof:

Let $\{A_\alpha\}$ be a collection of connected subspace of A.

Let P be a point of $\cap A_\alpha$

Let $Y = \cup A_\alpha$

T.P: Y is connected.

Suppose Y is not connected then Y has separation.

Let $Y = C \cup D$

where $C \neq \phi$, $D \neq \phi$, $C \cap D = \phi$

C and D are open in Y.

Now, $p \in Y$

$\Rightarrow p \in C$ (or) $p \in D$

Assume that $p \in C$

$\because$ each A_α is connected it must lie entirely in either C or D.

And it can't lie in D because it contains the point p of C.

Hence $A_\alpha \subset C$ for every α

$\cup A_\alpha \in C$.

i.e., $Y \subset C$

$\because D \neq \phi$

$\therefore$ Y is connected.

U.Q (S) (M.R) Thm 23.4

Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$ then B is also connected

Proof:

Let A be connected subspace of X &

Let $A \subset B \subset \overline{A}$.

T.P :- B is connected.

Suppose B is not connected then B have a separation

Then $B = C \cup D \longrightarrow (1)$

where $C \neq \phi$, $D \neq \phi$, $C \cap D = \phi$

C & D are open in B.

$\because$ A is connected,

by Lemma 23.2.

The set A must lie entirely in C or D

i.e., $A \subset C$ (or) $A \subset D$

If $A \subset C \Rightarrow \bar{A} \subset \bar{C}$

$B \subset \bar{C} \rightarrow$ (2) $(\because A \subset B \subset \bar{A})$

From (1) & (2)

$B \cap D = \phi$ $(\because \bar{C} \cap D = \phi)$.

Which is $\Rightarrow$ ⇐ to that D is not every subset of B

$\therefore$ B is connected.

Hence the proof

U.Q (X) R-5M

Thm 23.5

The image of a connected space under a Continuous map is connected.

Proof:

Let $f: X \rightarrow Y$ be a continuous map.

Let X be connected.

T.P:

The image space $Z = f(X)$ is connected.

$\because$ The map obtained from f by restricting its range to the space Z is also continuous.

Consider the Case of a continuous surjective map $g: X \rightarrow Z$

Suppose that Z is not connected then Z has a separation.

i.e., $Z = A \cup B$ where $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$

and A and B are open in set Z.

Then $g^{-1}(Z) = g^{-1}(A \cup B)$

$\Rightarrow X = g^{-1}(A) \cup g^{-1}(B)$

$\because g : X \to Z$ is continuous and A, B are open in set Z.

$\Rightarrow g^{-1}(A)$ & $g^{-1}(B)$ are open in X.

A and B are disjoint.

$g^{-1}(A)$ & $g^{-1}(B)$ are disjoint.

A and B are non-empty.

$g^{-1}(A)$ & $g^{-1}(B)$ are non-empty.

$\therefore g^{-1}(A)$ & $g^{-1}(B)$ from a Separation of X

Which is contradiction to X is Connected.

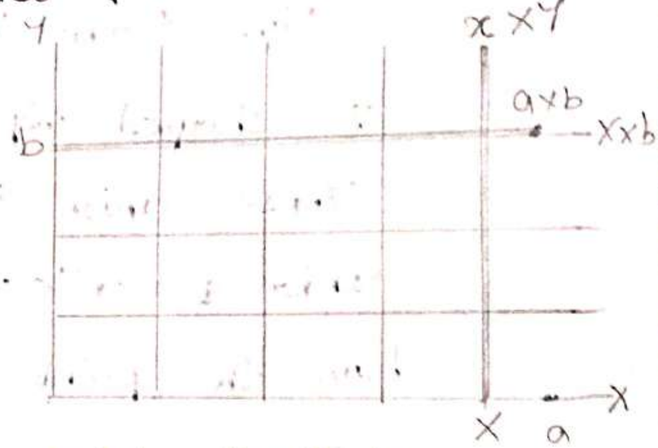
$\therefore Z$ is connected.

Hence the proof.

10 M R.U.

Thm 23.6

sst: A Finite Cartesian product of connected space is Connected



Induction method n=2

Proof:

First we prove the thm for the product of two connected spaces X and Y.

Now, consider the base point $A \times B$ in the product $X \times Y$.

Consider the horizontal line $X \times b$ is connected, homomorphic with X.

Also each vertical line $x \times Y$ is connected, being homomorphic with Y.

$f : X \to X \times b$ by $f(x) = X \times b$ is homomorphic.

But X is connected.

By thm 23.5

The image of connected is connected then

$X \times b$ is connected.

$|||^{ly}$ The vertical line $X \times Y$ is homomorphic with Y

$f : Y \to x \times Y$ by $f(y) = x \times Y$ is homomorphic

But Y is connected.

$\Rightarrow x \times Y$ is connected.

As a result each T- shaped Space

$T_x = (X \times b) \cup (x \times Y)$ is connected

T_x is the union of two connected Spaces, that have the point $x \times b$ in common.

Now form union $\bigcup_{x \in X} T_x$ of all these T- shaped spaces.

These union is connected because its the union of a collection of connected space that have the point $a \times b$ in Common.

$\therefore$ This union equals $X \times Y$, the shapes $X \times Y$ is connected.

The proof for any finite product of connected spaces follows by induction.

Let $X_1, X_2, \dots X_n$ be n - connected spaces and $X_1 \times X_2 \times X_3 \times \dots \times X_{n-1}$ is connected

T.P: $X_1 \times X_2 \times \cdots \times X_{n-1} \times X_n = (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$.

$\because$ $X_1 \times X_2 \times \cdots \times X_{n-1}$ and X_n are connected and we proved above that product of two connected space is connected.

$\Rightarrow$ $X_1 \times X_2 \times \cdots \times X_n$ each connected.

Thus, finite cartesian product of connected space is connected.

Hence the proof.

Sec 24:

U.Q Q 2M Connected subspace of the real line [Linear Continuum]:

A simply order set L having more than one elt is called a Linear continum if the following holds.

i) L has the least upper bound property

ii) If $x<y$, $\exists$ z s.t $x<z<y$.] CIA - II portion

Thm 24.1

10M Q U.Q If L is a linear continum in the order topology then L is connected and so are, intervals and rays in L.

Proof:

Let Y be convex subspace of L.

i.e for every pair of points (a, b) of in Y with $a<b$, the entire interval $[a,b]$ of points of L lies in Y.

T.P: If Y is convex subspace of L then Y is connected.

Suppose that Y is the union of disjoint non-empty sets A & B where A & B are open in Y.

$\therefore$ Y is not connected.

$\therefore$ $Y = A \cup B$ where $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$

Now choose $a \in A$, $b \in B$ and $a<b$

$\therefore$ $[a,b] \subset Y$ ($\because$ Y is convex).

let $A_0 = A \cap [a,b]$

$B_0 = B \cap [a,b]$

Then $A_0 \neq \phi$ [$\because a \in A_0$]

$B_0 \neq \phi$ [$\because b \in B_0$]

$A_0 \cap B_0 = \phi$ [$\because a \cap b = \phi$].

A_0 & B_0 are open in $[a,b]$ in the subspace topology which is same as the order topology.

Also $[a,b] = A_0 \cup B_0$

$\therefore$ A_0 & B_0 from a sep of $[a,b]$

upper bound

let $C = \sup A_0$

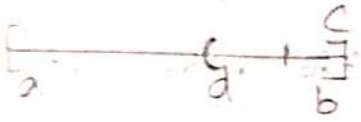
Now, we have to show $C \notin A_0$ & $C \notin B_0$.

Case (i)

Suppose that $C \in B_0$ then $C \neq a$.

So either $c = b$ (or) $a < c < b$.

In either case it follows that B_0 is open in $[a,b]$ that there is some interval of the form $[d,c]$ s.t $x_0 > d$ then $x_0 \in (d,c]$.



$\therefore x_0 \in B_0$

$\Rightarrow x_0 \in A_0 \cap B_0$

$\Rightarrow A_0 \cap B_0 \neq \phi$

$\Rightarrow\Leftarrow$ to the fact that $A_0 \cap B_0 = \phi$

when $a < c < b$.

Then the interval $[c,b]$ doesn't intersect A_0 because c is least upper bound of A_0

Then $[a,b] = [a,c] \cup (c,d]$

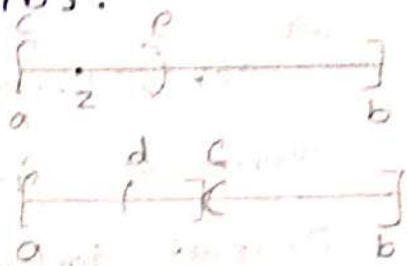
which doesn't intersect A_0

Again d is a least upper bound A_0 which is ... to the find c in LUB of A_0.

$\therefore c \notin B_0$.

Claim (ii)

Suppose $c \in A_0$ i.e. $c \in A_0 = A \cap [a,b]$.



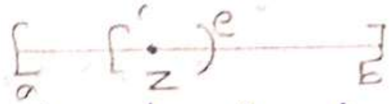
$\Rightarrow c \neq b$

$\Rightarrow c = a$ (or) $a < b < c$.

$\therefore A_0$ is open in $[a,b]$.

There is some interval $[c,e)$ contained in A_0.

By ordered topology property (ii) of linear continuum L we can choose a point z of L s.t $c < z < e$ then $z \in A_0$.



which is $\Rightarrow\Leftarrow$ to c is upper bound for A_0.

$\boxed{\therefore c \notin A_0}$

Hence for case (i) & (ii)

$c \notin A_0$ & $c \notin B_0$

$\Rightarrow c \notin A_0 \cup B_0$

But $c \in [a,b]$

which is $\Rightarrow\Leftarrow$ $[a,b] = A_0 \cup B_0$.

$\therefore$ The convex set Y is connected.

Hence L is connected interval & rays in L are connected.

Hence the proof.

UQ 5M Corollary

The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Proof :

$\because$ $\mathbb{R}$ is linear continuum.

By the Previous thm,

$\mathbb{R}$ is connected also intervals and rays in are connected.

UG
(X) R
5M

Thm : Intermediate Value theorem :

Sst:

Let $f: X \to Y$ be a continuous map, where X is a connected Space and Y is an ordered set in the ordered topology. If a and b are two points of X and if r is a point of Y lying b/w $f(a)$ and $f(b)$ then $\exists$ a point c of X such that $f(c) = r$. → intermed of X and

Proof :

Let $f: X \to Y$ is Continuous function where X is a connected space and Y is ordered set.

If $a, b \in X$ then $f(a), f(b) \in f(X) \subset Y$.

T.P: If $r \in Y$ where $f(a) < r < f(b)$ then $\exists$ a point c of X such that $f(c) = r$.

Suppose $f(c) \neq r$.

Then $f(c) < r$ (or) $r < f(c)$.

$f(c) \in (-\infty, r)$ (or) $f(c) \in (r, \infty)$

Define $A = f(X) \cap (-\infty, r)$, $\delta A \neq 0$ ($\because f(a) \in A$).

$B = f(X) \cap (r, \infty)$, $\delta B \neq 0$ ($\because f(b) \in B$)

and $A \cap B = \phi$ ($\because (-\infty, r) \cap (r, \infty) = \phi$)

Also A and B are open in the Subspace of $f(X)$

then $f(X) = A \cup B$

$\therefore f(X)$ has a separation

$\therefore f(X)$ is not connected.

Which is a $\Rightarrow\Leftarrow$ to the fact that the image of a connected space under a continuous map is connected

Hence if r is a point of Y lying between $f(a)$ and $f(b)$ then $\exists_1$ a point c of X such that $f(c) = r$.

Path connected: H.P.

Path:

Given Pts (x, y) x and y of the space X, a path in X from x to y is a continuous map of $f : [a,b] \to X$ of some closed interval in the real line into X such that $f(a) = x$ and $f(b) = y$.

A space X is said to be path connected if every pair of points of X can be joined by a path in X.

Result:

Every path connected space is connected.

Soln:

Let X be a path connected space.

T.P: X is connected.

Suppose X is not connected then X has separation

Let $X = A \cup B$ where $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$ and A and B are open in X.

Let $x \in A$, $y \in B$ $(\because A \neq \phi, B \neq \phi)$

$\therefore (x, y) \in X$

$\therefore X$ is a path connected...

It a path connecting the points x and y

i.e. x and $y \in A$ but $y \in B$.

$\therefore A \cap B \neq \phi$

which is $\Rightarrow\Leftarrow$ our assumption.

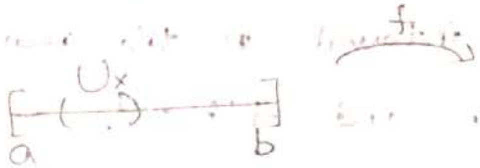
$\therefore X$ is connected.

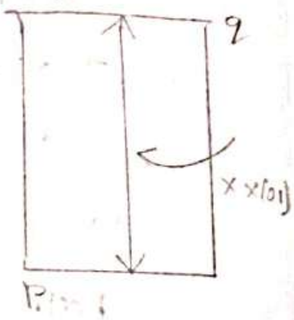
But converse is not true.

i.e. A connected space need not be path connected.

10M U.Q 5M Eg For Path connected

P.T Ordered squares I_0^2 is connected but not path connected.





Proof:

Since I_0^2 is a linear continuum.

I_0 is connected and product of Connected Space is connected.

$\therefore I_0^2$ is connected.

T.P : I_0^2 is not path connected.

Consider $P = 0 \times 0$ and $q = 1 \times 1$.

Suppose there is a path $f : [a,b] \to I_0^2$ joining p and q then we get a $\Rightarrow\Leftarrow$.

By intermediate value thm, the image set $y = f[a,b]$ must containing every point $x \times y$ of I_0^2.

$\therefore$ Their exist a point $x \times y$ such that $f(x \times y)$.

$\because$ Every point of I_o^2 lies b/w in $(0,0)$ and $(1,1)$

i.e. $(0,0) \leq (x \times y) \leq |x| \; \forall$

$\therefore$ For each $x \in I$ the set $U_x = f^{-1}(x \times (0,1))$

is a non empty subset of $[a,b]$.

By continuity, it is an open in $[a,b]$

Choose for each $x \in I_o$, a rational number $q_x \in U_x$.

$\because$ The sets U_x are disjoint, the map $h: x \to q_x$ is

an injective mapping of I into $\mathbb{Q}$ given by

$h(x) = q_x$

Which is a $\Rightarrow \Leftarrow$ to the fact the interval

I is uncounted.

$\because$ There doesn't exist a path in $I_o \times I_o$

~~Conta~~ connecting p & q.

I_o^2 is not path connected.

Sec 25

Components and Local Connectedness

components:

Given X, define an equivalence relation on X

by setting $x \sim y$ if there is a connected subspace

of X containing both x and y. The equivalence

classes are called the components (connected

components) of X.

U.Q
Thm 25.1

The components of X are connected disjoint

Subspace of X whose union is X such that

each non-empty connected subspace of X

intersects only one of them.

Proof:

Being equivalence classes, the components of X are disjoint and there union is disjoint

Let A be a non-empty connected subspace of X

If A intersects the components C_1 and C_2 of X say in points x_1 and x_2 respectively.

Then $x_1 \sim x_2$

First, we have to prove A intersects only one of the components.

By def

This can't happen unless $C_1 = C_2$.

$\therefore$ A intersect only one component of X.

Let C be a component of X.

T.P: C is connected.

Choose a point $x_0 \in C$ for each point x of C,

W.K.T $x_0 \sim x$

$\Rightarrow \exists$ a connected subspace A_x.

$\Rightarrow A_x$ containing C.

$\therefore C \bigcup_{x \in C} A_x$

$\because$ The subspace A_x are connected and have the point x_0 in common. there union is connected

Hence proved.

Path Components :

Given a topology space X, defined an equivalence relation on the space X by defining $\boxed{x \sim y}$ if there is a path in X from x to y. The equivalence classes are called path components of X.

Thm 25.2

The path connected component of X are path connected disjoint subspaces of X whose union is X. Such that each non-empty path connected subspace intersect only one of them.

Proof:

Note that each components of a space X is Closed in X.

∵ The closure of a connected subspace of X is connected.

If X has only finitely many components then each component is also open in X.

∵ Its Complement is finite union of closed set

In general the component of X need not be open in X.

In the path components of X, for they need be neither open not closed in X

Hence proved.

Locally connected :

(when a space connected)

A space X is said to be locally connected at x if for every neighbourhood U of x there is a connected neighbourhood be of x contained in U.

If X is locally connected at each of its point, its said locally connected.

2M U. Locally path connected

A space X is said to be locally path connected at x if for every neighbourhood U of x, there is a path connected neighbourhood V of x, contained in U. If X is locally path connected at each of its point of its point, then its said to be Locally path connected.

U.Q 10M R Thm 25.3

SST: A space X is locally connected iff for every open set U of X each component of U is open in X.

Proof :

Suppose that X is locally connected.

Let U be an open in X.

Let C be a component of U.

If x is a point of C, we can choose a connected neighbourhood V on x such that V contained U.

∵ V is connected.

It must lie entirely in the components C of U.

∴ C is open in X.

Conversely.

Suppose that components of open sets in X are open. Given a point x of X, and a nbhd U of x, let C be the components U containing x.

Now, C is connected,

∴ its open in X.

By hypothesis, X is locally connected at x

Hence the proof.

Thm 25.4.

5M A space X is locally path connected iff for every open set U of X, each Path component of U is open in X.

Proof: Previous above thm proof (change the words connected → Path connected)

The Relation b/w Path components and components.

V.Q 10M Thm: 25.5

If X is a topological space, each path components of X lies in a Components of X. If X is locally path connected then the components and the path components of X are the same.

Proof:

Let C be a component of X. Let x be a point of C.

Let P be the path component of X containing x.

Let x be a point of C

∴ P is connected, $P \subset C$.

To show that if X is locally path connected then $P = C$.

Suppose that $P \subsetneq C$ not equal to

Let Q denote the union of all the path component of X that are different from P and intersect C.

Each of them necessarily lies in C. so that

$C = P \cup Q$

Because, X is locally path connected each path component of X is open in X.

$\therefore$ P and Q are open in X.

This consistute a separation of C.

This contradicts the fact that C is connected.

$\therefore$ If X is locally path connected then the components and path components.

Hence the proof.

Additional Resource :

http://mathforum.org

http://ocw.mit.edu/ocwweb/Mathematics

http://www.opensource.org

http://en.wikipedia.org

Practice Questions:

Question Bank

Section – A

1.Define Connected Spaces

2.Define separation

3.Define linier continum

4.State Intermediate Value theorm.

5.Define unit Sphere.

6.Define Components.

7.Define path connected.

8.Define locally path connected.

Section – B

1.Prove that the union of a collection of connected subspaces of X that have a point in common is connected.

2.Prove that the components of X are connected disjoint subspaces of X whose Union is X,Such that each nonempty connected subspace of X intersects only one of them.

3.Prove that the image of a connected space under a continuous map is connected.

4.Prove that a space X is locally connected if and only if for every open set U of X, each component of U is open in X.

5.Prove that the space $I \times I$ in the order topology is connected but not in path connected.

5.If the sets C and D from a separation of X and if Y is a connected subspace of X, Prove that Y lies entirely within C or D.

6.Let A be a connected subset of X. If $A \subset B \subset \bar{A}$,Prove that B is also connected.

7.State and prove intermediate value theorem.

8.Prove that a finite Cartesian product of connected spaces is connected.

9.A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

10.P.T every path connected space is connected.

11.Prove that a space X is locally path connected if and only if for every open set U of X, each component of U is open in X.

Section – C

1.Prove that a finite Cartesian product of connected space is connected.

2.If X is a topological space, each path component of X lies in a component of X. If X is locally path connected , Prove that the components and the path Components of X are the same.

3.If L is a linear continuum in the order topology, prove that L is connected, and so are intervals and rays in L.

4.P.T the real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$

Recommended Text : James R. Munkres, Topology (2nd Edition) Pearson Education Pve. Ltd., Delhi-2002 (Third Indian Reprint)