MARUDHAR KESARI JAIN COLLEGE FOR WOMEN (AUTONOMOUS)
VANIYAMBADI
PG and Department of Mathematics

II M.Sc Mathematics – Semester - III

E-Notes (Study Material)

Core Course : Topology **Code: 23PMA33**
UNIT-II: Continuous functions: Continuous functions – the product topology – The metric topology.
Learning Objectives: To study Continuous functions: Continuous functions – the product topology – The metric topology.
Course Outcome: Understand continuity. Analyze and apply the topological concepts in Functional Analysis

Overview:

1. **Continuous Functions**:
   A function between two topological spaces is continuous if small changes in the input result in small changes in the output. In simple terms, the function doesn't "jump" or have any breaks.
2. **Product Topology**:
   The product topology helps define a topology on the product of two spaces. A function is continuous if it behaves continuously with respect to the topologies of the individual spaces in the product.
3. **Metric Topology**:
   The metric topology is a way of defining a topology using a distance function (called a metric). It tells us how close or far apart points are, and a function is continuous if small changes in the input lead to small changes in the output based on this distance.

This unit explains how to understand and work with continuous functions in different types of topological spaces.

UNIT - 2.

Continuous Function

Sec - 18 : Continuous Function.

Continuity of a Function :

Let X and Y be topological space. A Function $f : X \to Y$ is said to be continuous if for each open subsets B V of Y, then set $f^{-1}(V)$ is an open subset of X.

Note:

If the topology of the range space Y is given by a basis then to prove continuity of F its sufficient to show that the inverse image of every basis element is open.

R.U.Q 10M

Thm 18.1

Let X and Y be a topological space. Let $f : X \to Y$ then the following are equivalent :

i). f is continuous

ii). For every subset $A (X)$, one has $f(\bar{A}) \subset \overline{f(A)}$

V.Q 5M R iii). For every closed set B of Y the set $f^{-1}(B)$ is closed in X.

iv). For every $x \in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of X, $f(U) \subset V$.

If the condition in (iv) holds for a point 'x' we say that f is continuous at the point.

Proof:

i/. We show that

(i) ⇒ (ii) ⇒ (iii) ⇒ (i)

(i) ⇒ (iv) ⇒ (i)

Now we show that eqn ① ⇒ ②

Gn: f is continuous

T.P: For every subset A(x), one has

$f(\bar{A}) \subset \overline{f(A)}$.

Let $f: X \to Y$ is continuous.

Let AB be a subset of X

It is enough to show that if $x \in \bar{A}$ then

$f(x) \in \overline{f(A)}$.

Let V be a neighbourhood of f(x).

Then V is open in Y.

∵ f is continuous.

It follows that $f^{-1}(V)$ is open in X.

Also, $f(x) \in V$

$\Rightarrow x \in f^{-1}(V)$

∴ $f^{-1}(V)$ is a neighbourhood points of x.

∵ $x \in \bar{A}$ then $f^{-1}(V)$ must intersect A in

Some point Y.

i.e., $y \in f^{-1}(V) \cap A$

∴ $f(y) \in V \cap f(A)$

i.e. every neighbourhood of $f(x)$ intersect $f(A)$

$\Rightarrow f(x) \in \overline{f(A)}$

$\therefore f(\bar{A}) \subset \overline{f(A)}$ the

Next we have to prove,

(ii) $\Rightarrow$ (iii)

let $f(\bar{A}) \subset \overline{f(A)}$

Let B be a closed in Y.

T.P: $f^{-1}(B)$ is closed in X.

Let $A = f^{-1}(B)$

$\therefore$ It is enough to prove $A = \bar{A}$

But $A \subset \bar{A} \longrightarrow (1)$ ($\because$ by def closed set)

If $A = f^{-1}(B)$

$\Rightarrow f(A) = B$ and

$\overline{f(A)} \subset \bar{B} \longrightarrow (2)$

Let if $x \in \bar{A}$

$f(x) \in f(\bar{A})$

$\Rightarrow f(x) \in \overline{f(A)}$ ($\because f(\bar{A}) \subset \overline{f(A)}$)

$\Rightarrow f(x) \in \bar{B}$ ($\because$ by eqn (2))

$\Rightarrow f(x) \in B$ ($\because \bar{B} = B$, B is closed)

Then $x \in f^{-1}(B)$

i.e. $x \in A$

Thus $\bar{A} \subset A \longrightarrow (3)$

From eqn (1) and eqn (2)

$\bar{A} = A$

i.e., $A = f^{-1}(B)$ is closed in X.

(3) ⇒ (1)

Gn: If B is closed in Y then $f^{-1}(B)$ is closed in

T.P: f is continuous.

Let V be an open set of Y.

Then Y − V is closed in Y and Let B = Y − V.

⇒ V = Y − B

$f^{-1}(V) = f^{-1}(Y-B)$

$= f^{-1}(Y) - f^{-1}(B)$

$f^{-1}(V) = X - f^{-1}(B)$.

By hypothesis

$f^{-1}(B)$ is closed in X

∴ $X - f^{-1}(B)$ is open in X.

i.e., $f^{-1}(V)$ is open in X

∴ f is continuous

(1) ⇒ (4)

Gn: F is continuous

T.P: Each neighbourhood B of f(x), there is a neighbourhood U of x such that

$f(U) \subset V$.

Let $x \in X$ and Let V be a neighbourhood of $f(x)$ then the set $U = f^{-1}(V)$ is neighbourhood of x such that $f(U) \subset V$.

(4) ⇒ (1)

T.P : F is continuous.

Let V be an open set of Y and Let x be a Point of $f^{-1}(V)$.

Then $f(x) \in V$

By hypothesis

There is a neighbourhood U_x of x such that $f(U_x) \subset V$ then $U_x \subset f^{-1}(V)$.

⇒ $f^{-1}(V)$ can be ~~retun~~ written as the union of the open set U_x.

⇒ $f^{-1}(V)$ is open in X.

Thus F is continuous.

Hence the proof.

Homeomorphisms:

Let X and Y be topological spaces. Let $f: X \to Y$ be a bijection. If both the f and the inverse function $f^{-1}: Y \to X$ are continuous. Then f is called homeomorphisms.

Note:

The another way to define is to say that its bijective correspondence, $f: X \to Y$ such that $f(U)$ is open iff U is open

In any property of X expressed in terms the open set of X then such a property X is called a topological property of X.

injective 1-1

Sub : onto

bijec - 1-1 & onto

Def: Topological Imbedding:

Suppose that $f: X \to Y$ is an injective continuous map where X and Y are topological Space, Let Z be the image set f(X), consider has a subspace of Y then the function $f': X \to Z$ obtained by restricting by range of f is bijective.

If f' happen to be homeomorphism of X with Z then (the map $f: X \to Y$ is ~~tolog~~ topological imbedding (or) Imbedding of X in Y.

10M
P ② U-Q
⊗

Thm 18.2.

Rules for constructing Continuous Function

Let X, Y and Z be topological Spaces.

a) Constant Function:

If $f: X \to Y$ maps are of X into single point y_0 of Y then f is continuous,

b) Inclusion:

If A is a subspace of X, the inclusion function $j: A \to X$ is continuous, c). Composite :

If $f : X \to Y$ and $g : Y \to Z$ are continuous then the map $g_o f : X \to Z$ is continuous.

d). Restricting the domain :

If $f : X \to Y$ is continuous and if A is a subspace of X, then there restricted function $f/_A : A \to Y$ is continuous.

e). Restricting or Expanding the range :

Let $f : X \to Y$ be continuous. If Z is a sub-space of Y containing the image set of $f(X)$ then the function $g : X \to Z$ obtained by restricting the range, of a f is continuous.

If Z is a space having Y has a subspace then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

f). ~~Globa~~ Local Formulation of continuity :

U.Q The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f/_{U_\alpha}$ is continuous for each α.

Proof :

a) let $f : X \to Y$ be defined by $f(x) = y_0$, for every x in X.

Let V be open in Y.

If $Y_0 \in V$ then $f^{-1}(V) = X$ which is open

If $y_0 \notin V$ then $f^{-1}(V) = \phi$ which is open

In either case $f^{-1}(V)$ is open in X

$\therefore$ F is continuous.

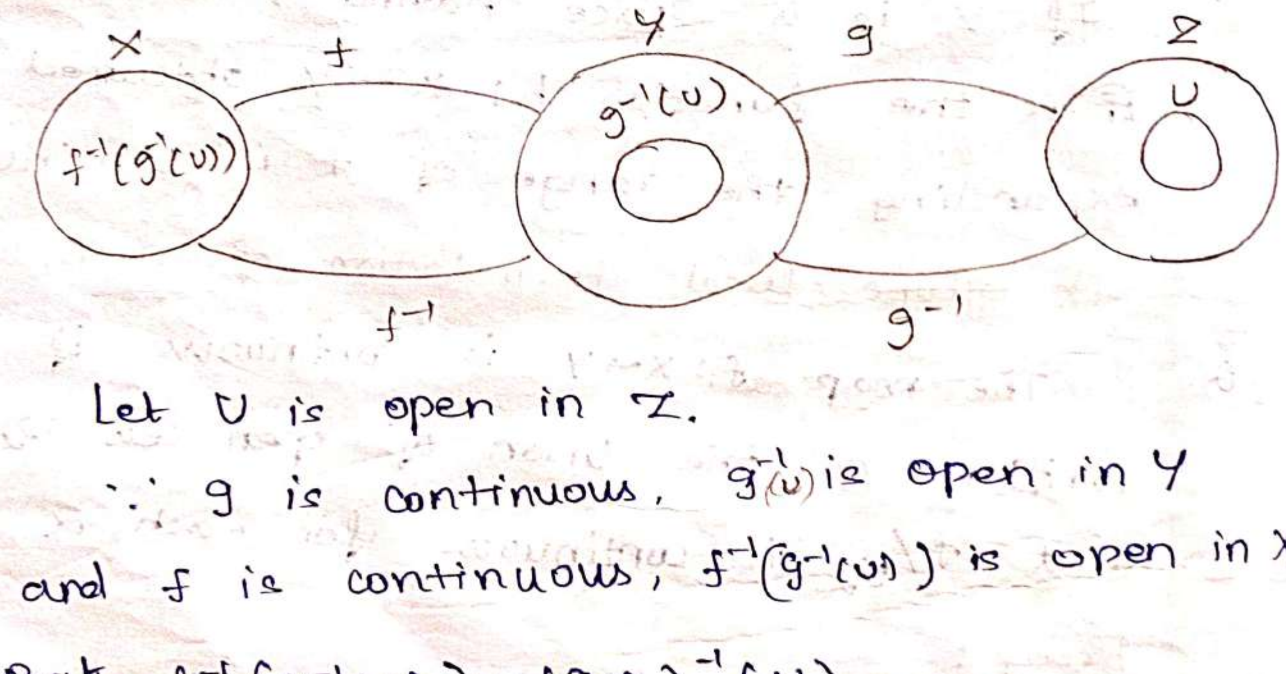
b) Let A is a subspaces of X and the inclusion function $j : A \to X$.

If U is open then $j^{-1}(U) = U \cap A$, which is open in A.

$\therefore$ By definition of subspace topology, j is continuous.

c) Gn: $f : X \to Y$ and $g : Y \to Z$ are continuous

T.P: $g \circ f : X \to Z$



Let U is open in Z.

$\therefore$ g is continuous, $g^{-1}(U)$ is open in Y

and f is continuous, $f^{-1}(g^{-1}(U))$ is open in X.

But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$

$\therefore$ $g \circ f$ is continuous.

d) Gn: $f : X \to Y$ is continuous and A is a subspace of X

T.P : $f/_A : A \to Y$ is continuous.

W.K.T $j : A \to X$ be inclusion function.

Then

$f/_A : f \circ j : A \to Y$.

$\because$ f and j are both are continuous and

the composite function is continuous.

We have $f/_A$ is continuous.

---

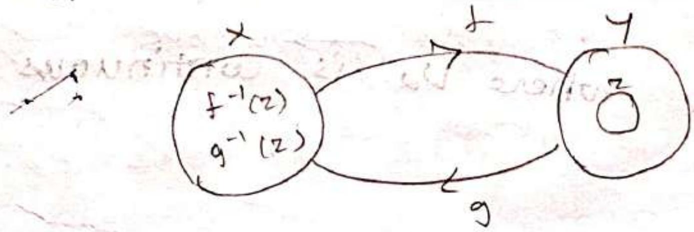
e.g. Gn: $f : X \to Y$ be continuous and

Let $f(x) \subset Z \subset Y$

T.P: $g : X \to Z$ obtained from f is continuous.

Let B be open in Z.

Then $B = Z \cap U$ for some open set $U(Y)$.



$\therefore g^{-1}(B) = g^{-1}(Z \cap U)$

$= g^{-1}(Z) \cap g^{-1}(U)$

$g^{-1}(B) = g^{-1}(U)$

$= f^{-1}(U)$ [$\because$ Z contains the entire image set $f(X)$].

$\because$ f is continuous and U is open in Y

then $f^{-1}(U)$ is open in X

$\Rightarrow g^{-1}(B)$ is in open in X

$\therefore$ g is continuous.

Now, if Z is a space having Y has a subspace.

i.e. $Z \supset Y$

Then to prove $h: X \to Y$ obtained by ~~expacting~~ expanding the range of f is continuous.

Let inclusion map $j: Y \to Z$, and we know $f: X \to Y$.

Then $f \circ j = h: X \to Z$

$\because$ f and j are continuous and the composite function of continuous

$\therefore$ h is continuous

~~$\therefore$ f is continuous~~

Q. Gn: $X = \cup U_\alpha$ where U_α is open such that f/U_α is continuous for each α.

T.P: $f: X \to Y$ is continuous.

Let V be open set in Y.

$\because$ f/U_α is continuous,

$(f/U_\alpha)^{-1}(V)$ is open in U_α and

Hence open in X.

But $(f/U_\alpha)^{-1}(V) = f^{-1}(V) \cap U_\alpha \longrightarrow$ (1)

Because the both expression represent the set of those points

x lying in U_α for which $f(x) \in V$.

And $f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha)$

$= \bigcup_\alpha ((f/U_\alpha)^{-1}(V))$ [$\because$ by (1)]

By hypothesis,

RHS is open in X, so that $f^{-1}(V)$ is open in X.

Hence f is continuous.

Hence the proof.

Q.Q (*) 5M

Thm 18.3

[The Pasting Lemma]

St:

Let $X = A \cup B$, where A and B are closed in X. Let $f: A \to Y$ & $g: B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combines to gives a continuous fn $h: X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$

Proof:

[⟹]

Let C be a closed subset of Y.

T.P: h is continuous.

It is enough to show that $h^{-1}(C)$ is closed in X.

Gn: $X = A \cup B$, where A & B are closed in X and $f: A \to Y$; $g: B \to Y$ are continuous.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \longrightarrow$ (1)

$\because f$ is continuous, $f^{-1}(C)$ is closed in A &

hence closed in X.

$|||^{ly}$ g is continuous. $g^{-1}(C)$ is closed in A &

hence closed in X.

$\therefore f^{-1}(C) \cup g^{-1}(C)$ is closed in X.

Hence by ①, $h^{-1}(C)$ is closed in X.

Thus h is continuous

Hence proved.

Thm 18.4 [Maps into products]

Sst:

Let $f: A \to X \times Y$ be given by the eqn $f(a) = (f_1(a), f_2(a))$

Then f is continuous iff the function $f_1: A \to X$ and

$f_2: A \to Y$ are continuous. The maps f_1 & f_2 are

called the Co-ordinates function of f.

Proof:

Gn: f is continuous.

T.P: $f_1: A \to X$ & $f_2: A \to Y$ are continuous

Let $\pi_1: X \times Y \to X$ & $\pi_2: X \times Y \to Y$ be projections

onto the 1st and 2nd factor respectively.

These maps are continuous.

Then $\pi_1^{-1}(U) = U \times Y$ be the open set in $X \times Y$

for any open set U in X and

$\pi_2^{-1}(V) = X \times V$ be the open set in $X \times Y$ for any open set V in Y, $\forall a \in A$.

Let $f_1(a) = \pi_1(f(a))$

$f_2(a) = \pi_2(f(a))$

$\Rightarrow f_1 = \pi_1 \circ f$ & $f_2 = \pi_2 \circ f$

$\therefore$ f is continuous and π_1 & π_2 both are continuous and by composite of continuous function and [$\therefore$ continuous] next line.

$\Rightarrow f_1$ & f_2 are continuous.

[$\Leftarrow$]

Gn: f_1 & f_2 are continuous.

T.P: $f: A \to X \times Y$ is continuous.

Let $U \times V$ be any basis open set in $X \times Y$ then to show $f^{-1}(U \times V)$ is open.

Let $a \in f^{-1}(U \times V) \Leftrightarrow f(a) \in U \times V$.

$\Leftrightarrow f_1(a) \in U$ & $f_2(a) \in V$

$\Leftrightarrow a \in f_1^{-1}(U)$ & $a \in f_2^{-1}(V)$

$\Leftrightarrow a \in f_1^{-1}(U) \cap f_2^{-1}(V)$

$\therefore f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V) \longrightarrow (1)$

$\because f_1$ & f_2 are continuous.

$f_1^{-1}(U)$ & $f_2^{-1}(V)$ are open in A

$\therefore$ By (1) $f^{-1}(U \times V)$ is open.

$\therefore f$ is continuous.

H.P.

Sec-19. Def:

Product Topology : (J - tuple)

Let J be an index set given a set X, we defined a J-tuple of element of X to be a function $x : J \to X$. If α is an element of J then the value of x at α by x_α rather than $x(\alpha)$ called the α^{th} - co-ordinates of x.

We denote the function x itself by the symbol $(x_\alpha)_{\alpha \in J}$.

Which is closed, we denoted the set of all J-tuple of element of X by X^J.

Cartesian product :

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Let $X = \bigcup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family is denoted by $\prod_{\alpha \in J} A_\alpha$ is defined to be the set of all J-tuple $(x_\alpha)_{\alpha \in J}$ of element of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

i.e., It is the set of all function $x : J \to \bigcup_{\alpha \in J} A_\alpha$ such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

U.Q 2M

24
U.Q

Box Topology:

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

Let a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$, the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$, U_α is open in X_α, for each $\alpha \in J$.

(The topology generated by this basis is called the Box topology.)

Projection Mapping:

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its β^{th} - Co-ordinates, $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$ is called the projection mapping associated with the index β.

Product Topology:

Let $\mathcal{S}_\beta$ denote the collection.

$S_\beta = \{\pi_\beta^{-1}(U_\beta) / U_\beta$ open in $X_\beta\}$ and Let $\mathcal{S}$ denote $\mathcal{S} \neq \bigcup_\beta$ the union this collection

$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$. (The topology generated by the Subbasis $\mathcal{S}$ is called the product topology.

In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.)

Thm 19.1 (comparision of the Box and product Topologies)

The Box topology on product $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open is X_α for each α and U_α equals X_α except for finitely many values of α.

Note:

i) For Finite product $\prod_{\alpha=1}^{n} X_\alpha$, the box topology and the product topology are same.

ii) Box product is finite then the product topology.

Thm 19.2.

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all subsets of all the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$ for each α, then its basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

Thm 19.3

Let A_α be a subspace of X_α, for each $\alpha \in J$. Then product A_α is a subspace of product X_α. If both products are given the box topology or if both products are given the product topology

Thm 19.4.

If each space X_α is a hausdroff space then $\prod X_\alpha$ is a hausdroff space in both the box and product topologies.

Proof:

Gin: Each X_α is a hausdroff space $\forall \alpha \in J$.

T.P: $\prod_{\alpha \in J} X_\alpha$ is hausdroff.

Let $x \neq y \in \prod_{\alpha \in J} X_\alpha$

Where $X = (x_\alpha)_{\alpha \in J}$ and $Y = (y_\alpha)_{\alpha \in J}$.

Without loss of generality we asume $x_\alpha \neq y_\alpha \in X_\alpha$

X_α is hausdroff, $\exists$ two disjoint neighbourhood U_α & V_α of x_α & y_α respectively in X_α.

$\pi^{-1}(U_\alpha)$ and $\pi^{-1}(V_\alpha)$ are the required disjoint neighbourhood of x and y in product $\alpha \in J$.

$\prod_{\alpha \in J} X$, with respect to both box and Product topology.

$\therefore \prod_{\alpha \in J} X_\alpha$ is hausdroff with respect to both box and Product topology

T. Q.Q SMF

Thm 19.5

Let $\{X_\alpha\}$ be an indexed family of spaces. Let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product are the box topology then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof:

Let $x = (x_\alpha)_{\alpha \in J} \in \prod \overline{A_\alpha}$

T.P: $x \in \overline{\prod A_\alpha}$

$\because (x_\alpha)_{\alpha \in J} \in \prod \overline{A_\alpha}$.

$\Rightarrow x_\alpha \in \overline{A_\alpha} \; \forall \alpha \in J$.

Let $U = \prod_{\alpha \in J} U_\alpha$ be a basis element either in box or product topology containing x.

$\therefore U_\alpha$ is a neighbourhood of x_α.

$\because x_\alpha \in \overline{A_\alpha}$, $U_\alpha \cap A_\alpha \neq \phi$, $\forall \alpha$.

($\because x \in \overline{A}$ which $\Leftrightarrow$ iff $U \cap A \neq \phi$).

Let $y_\alpha \in U_\alpha \cap A_\alpha$ for each α.

Let $y = (y_\alpha)_{\alpha \in J}$ then,

$$y \in \left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} A_\alpha\right)$$

i.e. $U \cap \prod_{\alpha \in J} A_\alpha \neq \phi$.

$[\because x \in \overline{A} \Leftrightarrow U \cap A \neq \phi]$

$\therefore x \in \overline{\prod_{\alpha \in J} A_\alpha}$

$\therefore \prod_{\alpha \in J} \overline{A_\alpha} \subset \overline{\prod_{\alpha \in J} A_\alpha} \longrightarrow (1)$

Conversely,

T.P: $\overline{\prod_{\alpha \in J} A_\alpha} \subset \prod_{\alpha \in J} \overline{A_\alpha}$

Let $x \in \overline{\prod_{\alpha \in J} A_\alpha}$, T.P: $x \in \prod_{\alpha \in J} \overline{A_\alpha}$

It is enough to prove $x_\beta \in \overline{A_\beta}$ for any given index β

Let V_β be an arbitrary open set of X_β containing x_β.

$\therefore \pi_\beta^{-1}(V_\beta)$ is an open set containing πX_β in both box and product topology.

$\because x \in \overline{\pi A_\alpha}$, $\pi_\beta^{-1}(V_\beta) \cap \prod_{\alpha \in J} (A_\alpha) \neq \phi$

Let $y \in \pi_\beta^{-1}(V_\beta) \cap \prod_{\alpha \in J} A_\alpha$.

$\Rightarrow y \in \pi_\beta^{-1}(V_\beta)$ and $y \in \prod_{\alpha \in J} A_\alpha$

$\Rightarrow y_\beta \in V_\beta$ and $y_\beta \in A_\beta$

$y_\beta \in V_\beta \cap A_\beta$

$\therefore V_\beta \cap A_\beta \neq \phi$

$\Rightarrow x_\beta \in \overline{A_\beta}$

i.e. $x \in \prod_{\alpha \in J} \overline{A_\alpha}$

$\therefore \overline{\prod_{\alpha \in J} A_\alpha} \subset \prod_{\alpha \in J} \overline{A_\alpha} \longrightarrow$ (2)

From (1) & (2)

$$\overline{\prod_{\alpha \in J} A_\alpha} = \prod_{\alpha \in J} \overline{A_\alpha}$$

Hence Proved.

Thm 19.6.

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the eqn $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each α.

Let πX_α have the product topology.

Then the function f is continuous iff each function f_α is continuous.

Proof:

[$\Rightarrow$]

Assume that $f: A \to \prod X_\alpha$ is continuous.

T.P: each function f_α is continuous.

Consider the β^{th} Projection map.

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$$

The function π_β is continuous.

for if U_β is open in X_β, then $\pi_\beta^{-1}(U_\beta)$ is open in $\prod_{\alpha \in J} X_\alpha$ w.r to product topology being a subbasis element.

Also $f_\beta = \pi_\beta \circ f$. (composite)

for f_β and $\pi_\beta \circ f$ are function from A to X_β

since, f and π_β are continuous and by composite of continuous function is continuous.

$\pi_\beta \circ f$ is continuous.

i.e., f_β is continuous $\forall \beta \in J$.

[$\Leftarrow$]

Conversely,

Suppose that each co-ordinate function

$f_\beta : A \to X_\beta$ is continuous.

T.P: $f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Consider any subbasis element $\pi_\beta^{-1}(U_\beta)$ in the Product space $\prod_{\alpha \in J} X_\alpha$ where U_β is open in X_β.

i.e. $f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right)$ is open in A.

Consider,

$$f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = (f^{-1} \circ \pi_\beta^{-1}) U_\beta$$
$$= (\pi_\beta \circ f)^{-1}(U_\beta)$$
$$= f_\beta^{-1}(U_\beta) \quad (\because (\pi_\beta \circ f) = f_\beta)$$

$\because$ U_β is open in X_β and since

$f_\beta : A \to X_\beta$ is continuous.

$\therefore f_\beta^{-1}(U_\beta)$ is open in A.

i.e. $f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right)$ is open in A.

$\therefore f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Hence proved.

Note:
The above thm is true only for product topology it is not true for box topology.

Eg:
Prove by an example that a function continuous on R^ω with r to product topology need not be continuous on R^ω with box topology.

Proof:
Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself.

$$\therefore \mathbb{R}^\omega = \prod_{n \in Z_+} X_n \text{ where } X_n = \mathbb{R} \text{ for each } n.$$

Define a function $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$ and the n^{th} co-ordinate function of f is the function $f_n(t) = t$.

each of the co-ordinate function $f_n : \mathbb{R} \to \mathbb{R}$ is continuous

∴ The function f is continuous if $\mathbb{R}^\omega$ is given in the product topology.

T.P: f is not continuous if $\mathbb{R}^\omega$ is given the box topology.

Consider the basis elements

$B = (-1, 1) \times (-1/2, +1/2) \times (-1/3, 1/3) \times$ for the box topology.

We have to P.T $f^{-1}(B)$ is not open in $\mathbb{R}$

Suppose $f^{-1}(B)$ is open in $\mathbb{R}$, then we can find same interval $(-\delta, \delta)$ about the point zero such that $0 \in (-\delta, \delta) \subset f^{-1}(B)$.

$$\Rightarrow f(-\delta, \delta) \subset B$$

$$\therefore \pi_n (f(-\delta, \delta)) \subset \pi_n(B)$$

$$(\pi_n \circ f)(-\delta, \delta) \subset (-1/n, 1/n)$$

$$\text{i.e.,} \ f_n(-\delta, \delta) \subset (-1/n, 1/n), \ \forall n.$$

which is impossible, our assumption is wrong.

∴ $f^{-1}(B)$ is not open in $\mathbb{R}$.

∴ f is not continuous in $\mathbb{R}^\omega$ is given the box topology.

Sec: 20. The Metric Topology

Metric :

A metric on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties.

i) $d(x,y) \geq 0$ $\forall$ $x, y \in X$; equality holds iff $x = y$

ii) $d(x,y) = d(y,x)$ $\forall$ $x, y \in X$

iii) $d(x,y) + d(y,z) \geq d(x,z)$ $\forall$ $x, y \in X$

(triangular inequality)

Note: ϵ - ball centered at x:

[Given a metric d on X, the number $d(x,y)$ is called the distance between x and y in the metric 'd'.

Def: ϵ-ball centered at x:

Given $\epsilon > 0$ the set $B_d(x, \epsilon) = \{y / d(x,y) < \epsilon\}$. of all points 'y' whose distance from x is less than ϵ. It is called the ϵ- ball centered at x.]

Metric topology:

If 'd' is a metric on the set X, then the collection of all ϵ balls $B_d(x, \epsilon)$ for $x \in X$, $\epsilon > 0$ is a basis for a topology on X, called the metric topology induced by d.

Result:

Let d be a metric on a set X. Then T.P.

$\mathcal{B} = \{B_d(x, \epsilon) / x \in X$ & $\epsilon > 0\}$ is a basis.

Proof:

$\because$ $x \in B_d(x, \epsilon)$ for any $\epsilon > 0$ the first condition for the basis is trivial.

Before checking the second condition for a basis we show that " if $y \in B_d(x, \epsilon)$ then $\exists$ $\delta > 0$ such that $B_d(y, \delta) \subset B_d(x, \epsilon)$"

Def $\delta = \epsilon - d(x,y)$

Then $B_d(y,\delta) \subset B_d(x,\epsilon)$

for if $z \in B_d(y,\delta)$

Then $d(y,z) < \epsilon - d(x,y)$

$d(x,y) + d(y,z) < \epsilon.$

But $d(x,z) \leq d(x,y) + d(y,z) < \epsilon.$

i.e., $d(x,z) < \epsilon$

$\therefore z \in B_d(x,\epsilon)$

$\therefore B_d(y,\delta) \subset B_d(x,\epsilon)$

Let $x \in B_1 \cap B_2$ where B_1 & $B_2 \in \mathcal{B}$

$\therefore B_1 = B_d(x,\delta_1)$ and

$B_2 = B_d(x,\delta_2)$.

Let $\delta = \min$ of (δ_1, δ_2).

$\therefore B_d(x,\delta) \subset B_d(x,\delta_1) \cap B_d(x,\delta_2)$

i.e., $B_d(x,\delta) \subset B_1 \cap B_2$

Take $B_d(x,\delta) = B_3$

$\therefore B_3 \in \mathcal{B}$

Also $x \in B_3 \subset B_1 \cap B_2$.

$\therefore$ The 2nd Condition for a basis holds. $\therefore \mathcal{B}$ is a basis.

Hence the proof.

U.Q If $d: X \times X \to \mathbb{R}$. defined by $d(x,y) = |x-y|$ check whether it is metric or not.

Result:

A set V is open in the metric topology induced by d′ iff for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Eg:

1). Given a set X, define.

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Here d is a discrete metric. It induces a discrete topology.

2). The standard metric on the real number $\mathbb{R}$ is defined by the eqn $d(x,y) = |x-y|$.

It is easy to check that 'd' is a metric the topology it induces is the same as the order topology.

each basis elt (a,b) for the order topology is a basis elt for the metric topology, indeed.

$(a, b) = B(x, \epsilon)$

where $x = \frac{(a+b)}{2}$ and $\epsilon = \frac{(b-a)}{2}$. and

Conversely, each ϵ ball $B(x, \epsilon)$ equals an open interval $(x-\epsilon, x+\epsilon)$.

Def:

Metrizable:

If X is a topological space, X is said to be metrizable if $\exists$ a metric d on the set X that induces the topology of X.

U.Q. 2M (8) Metric Space:

A metric space ~~of~~ is a metrizable space X together with a specific metric d that gives the topology of X.

Bounded :

Let X be a metric space with metric 'd'. A subset 'A' of X is said to be bounded if there is some number M such that $d(a_1, a_2) \leq M$, for every pair a_1, a_2 of points of A.

Diameter :

If A is bounded and non empty the diameter of A is defined to be the number.

$$\text{diam } A = \sup \{ d(a_1, a_2) / a_1, a_2 \in A \}.$$

Note:

Boundedness of a set is not a topological property, for it depends on the particular metric 'd' that is used for X.

Thm 20.1.

Let X be a metric space with metric 'd'. Define $\bar{d} : X \times X \to \mathbb{R}$ by the eqn $\bar{d}(x,y) = \min \{ d(x,y), 1 \}$

Then $\bar{d}$ is a metric that induces the same topology as d. [The metric $\bar{d}$ is called the standard bounded metric corresponding to d].

Proof:

CLAIM: $\bar{d}$ is metric.

Checking the First two conditions for a metric is trivial.

Let us check the triangular inequality

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$$

Case (i):

If either $d(x,y) \geq 1$ (or) $d(y,z) \geq 1$.

Then the right side of this inequality is atleast 1.

i.e; $\bar{d}(x,y) + \bar{d}(y,z) \geq 1 \longrightarrow (1)$

$\because$ the left side is atmost 1.

i.e; $\bar{d}(x,z) \leq 1 \longrightarrow (2)$

Combining (1) & (2)

$$\bar{d}(x,z) \leq 1 \leq \bar{d}(x,y) + \bar{d}(y,z).$$

i.e; $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$.

Case (ii).

If $d(x,y) < 1$ and $d(y,z) < 1$

then
$$\bar{d}(x,y) = d(x,y) \text{ and}$$
$$\bar{d}(y,z) = d(y,z).$$

Now,

$$\bar{d}(x,z) \leq d(x,z) \quad (\because \text{by def of } \bar{d})$$
$$\leq d(x,y) + d(y,z)$$
$$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z).$$

$\therefore$ triangle inequality holds for $\bar{d}$.

T.P: d' topology and $\bar{d}$ topology are the same

Let $x \in R$ and $\epsilon > 0$

then $B_{\bar{d}}(x,\epsilon)$ is a basis elt containing x.

CLAIM:

$$B_d(x,\epsilon) \subset B_{\bar{d}}(x,\epsilon)$$

Let $y \in B_d(x,\epsilon) \Rightarrow d(x,y) < \epsilon$

$\Rightarrow \bar{d}(x,y) < \epsilon \quad [\because \bar{d}(x,y) \leq d(x,y)]$

$\Rightarrow y \in B_{\bar{d}}(x, \epsilon)$

$\therefore B_d(x, \epsilon) \subset B_{\bar{d}}(x, \epsilon)$

$\therefore$ d - topology $\subseteq$ $\bar{d}$ topology $\rightarrow$ (3)

let $x \in R$ and $\epsilon > 0$, where $\epsilon > 1$ then $B_d(x, \epsilon)$ is a basis elt containing x in the d - topology.

CLAIM :

$$B_{\bar{d}}(x, \epsilon) \subset B_d(x, \epsilon)$$

let $y \in B_{\bar{d}}(x, \epsilon) \Rightarrow \bar{d}(x, y) < \epsilon < 1$

$\Rightarrow \bar{d}(x, y) < 1$

$\Rightarrow d(x, y) < \epsilon$

$\Rightarrow y \in B_d(x, \epsilon)$

$B_{\bar{d}}(x, \epsilon) \subset B_d(x, \epsilon)$

$\therefore$ $\bar{d}$ topology $\supseteq$ d - topology

From (3) & (4) we get

$\bar{d}$ topology and d - topology are same.

Hence proved.

Def : Euclidean metric :

Norm :

Given $X = (x_1, x_2 \ldots x_n)$ in $\mathbb{R}^n$. The norm of X is defined by the eqn

$$\|x\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$

$$= \sqrt{(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

Euclidean metric

Given $\mathbf{x} = (x_1, x_2 \dots x_n)$ in $\mathbb{R}^n$. Then the euclidean metric d on $\mathbb{R}^n$ is defined by the eqn

$$d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|$$

$$= \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2 \right]^{1/2}$$

Square metric:

Given $\mathbf{x} = (x_1, x_2 \dots x_n)$. The square metric ρ is defined by the eqn

$$\rho(\mathbf{x}, \mathbf{y}) = \max \{ |x_1 - y_1|, \dots |x_n - y_n| \}$$

Lemma: 20.2

Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they inducdes, respectively. Then $\mathcal{T}'$ is finer that $\mathcal{T}$ iff for each x in X and each $\epsilon > 0$ $\exists$ a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Proof:

Let $\mathcal{T}'$ is finer than $\mathcal{T}$

Let $x \in X$ and $\epsilon > 0$ then $B_d(x, \epsilon)$ is a basis element w.r to the topology $\mathcal{T}$.

$\therefore B_d(x, \epsilon)$ is open w.r to $\mathcal{T}$.

$\because \mathcal{T}' \supset \mathcal{T}$, $B_d(x, \epsilon)$ is open w.r to $\mathcal{T}'$ then the metric d' induces the topology $\mathcal{T}'$, $\exists$ a basis element B' for the topology $\mathcal{T}'$ such that $x \in B' \subset B_d(x, \epsilon)$ where B' is in the form $B_{d'}(x, \delta)$.

$\therefore$ for each x in X and $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$

$\Leftarrow$ T.P: $\mathcal{J} \subset \mathcal{J}'$.

Let the $\delta - \epsilon$ condition holds.

Let $U \in \mathcal{J}$ containing x.

$\therefore U$ is neighbourhood of x w.r to $\mathcal{J}$.

$\because$ the metric 'd' induces the topology $\mathcal{J}$, $\exists$ a basis element $B_d(x, \epsilon) \subset U$.

By using $\delta - \epsilon$ condition $\exists$ $\delta > 0$ such that

$B_{d'}(x, \delta) \subset B_d(x, \epsilon) \subset U$.

$\therefore B_{d'}(x, \delta) \subset U$

where $B_{d'}(x, \delta)$ is a basis element w.r to $\mathcal{J}'$

$\therefore$ U is open w.r to $\mathcal{J}'$

$\Rightarrow U \in \mathcal{J}'$

$\therefore \mathcal{J} \subset \mathcal{J}'$

i.e. $\mathcal{J}'$ is finer than $\mathcal{J}$.

Hence the proof.

U.Q 10M

Thm : 20.3

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are same as the product topology on $\mathbb{R}^n$.

Proof

Let $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$ be two points on $\mathbb{R}^n$.

First we shall prove the inequality

$$\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y).$$

The metric d and ρ are defined as follows.

$$d(x,y) = \left[(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots (x_n-y_n)^2\right]^{1/2}.$$

$$\rho(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}.$$

Let $|x_i - y_i|^2 \le \sum_{i=1}^{n} |x_i - y_i|^2$

$$\le \sum_{i=1}^{n} (x_i - y_i)^2$$

$$\le (d(x,y))^2.$$

$\Rightarrow |x_i - y_i| \le d(x,y);\ i = 1, 2, \ldots, n.$

$\therefore \max\{|x_i - y_i| \mid i = 1, 2 \ldots n\} \le d(x,y)$

$$\rho(x,y) \le d(x,y) \longrightarrow ①$$

and Let $|x_i - y_i| \le \max\{|x_i - y_i| / i = 1, 2 \ldots n\}$

$$|x_i - y_i| \le \rho(x,y)$$

$$|x_i - y_i|^2 \le [\rho(x,y)]^2,\quad i = 1, 2 \ldots n$$

$$(x_i - y_i)^2 \le n[\rho(x,y)]^2.$$

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le n[\rho(x,y)]^2.$$

$$[d(x,y)]^2 \le n[\rho(x,y)]^2.$$

$$\Rightarrow d(x,y) \le \sqrt{n}\,\rho(x,y) \longrightarrow ②$$

Combining ① & ② we get

$$\rho(x,y) \le d(x,y) \le \sqrt{n}\,.\,\rho(x,y).$$

The First inequality shows that

$B_d(x, \epsilon) \subset B_\rho(x, \epsilon) \; \forall \; x$ and ϵ

$\because$ If $d(x,y) < \epsilon$ ~~$\rho(x,y)$~~ & $\rho(x,y) < \epsilon$. Also

||| ly

The second inequality shows that

$B_\rho(x, \epsilon/\sqrt{n}) \subset B_d(x, \epsilon) \; \forall \; x$ and ϵ

by the above Lemma,

That the two metric topology are the same.

Now we show that the product topology is the same as that given by the metric ρ.

T.P:

ρ - topology $\supset$ product topology.

Let $X \in \mathbb{R}^n$, where $x = (x_1, x_2 \ldots x_n)$.

Let $B = (a_1, b_1) \times \ldots \times (a_n, b_n)$ be a basis element for the product topology containing the element X.

$\therefore X_i \in (a_i, b_i) \; \forall \; i = 1, 2 \ldots n$

Choose $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$

$\therefore \prod (x_i - \epsilon_i, x_i + \epsilon_i) \subset \prod_{i=1}^{n} (a_i, b_i)$

Take $\prod_n (x_i - \epsilon, x_i + \epsilon) \subset \prod_{i=1}^{n} (a_i, b_i) = B$.

$\therefore \prod_n (x_i - \epsilon, x_i + \epsilon) \subset B$.

CLAIM:

$B_\rho(X, \epsilon) \subset B$.

Let $y \in B_\rho(X, \epsilon)$

$\Rightarrow \rho(x, y) < \epsilon$.

$\Rightarrow \max\{|x_1-y_1|, |x_2-y_2|, \ldots |x_n-y_n|\} < \epsilon$

$\Rightarrow |x_i-y_i| < \epsilon \;\forall\; i = 1, 2 \ldots n$

$\Rightarrow |y_i - x_i| < \epsilon \;\forall\; i = 1, 2 \ldots n$

$-\epsilon < y_i - x_i < \epsilon, \; i = 1, 2 \ldots n$

$x_i - \epsilon < y_i < x_i + \epsilon, \; i = 1, 2 \ldots n$

$\Rightarrow y_i \in (x_i - \epsilon, x_i + \epsilon)$

$\Rightarrow y \in (x_i - \epsilon, x_i + \epsilon) \subset B$

$\therefore y \in B$

$\therefore y \in B_\rho(x, \epsilon) \Rightarrow y \in B$

$\therefore B_\rho(x, \epsilon) \subset B$

$\therefore$ P topology $\supset$ product topology.

Conversely,

Let $B_\rho(x, \epsilon)$ be a basis element for the ρ-topology.

Given the element $y \in B_\rho(x, \epsilon)$, we need to find a basis element B for the product topology such that $y \in B \subset B_\rho(X, \epsilon)$

But this is trivial,

for $B_\rho(x, \epsilon) = (x_1-\epsilon, x_1+\epsilon) \times \ldots \times (x_n-\epsilon, x_n+\epsilon)$

is itb if a basis element for the product topology

∴ Product topology ⊃ ρ topology.

∴ ρ topology and product topology on $\mathbb{R}^n$ are the same.

Hence d- topology and the ρ- topology are the same as the product topology on $\mathbb{R}^n$.

Hence proved.

2M uniform topology:

Given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$. Let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the eqn

$$\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in J\}.$$

Where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. The metric $\bar{\rho}$ is called the uniform metric on $\mathbb{R}^J$, and the topology it induces is called the uniform topology.

Thm 20.4

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all difference if J is infinite.

Proof:

T.P: Box topology ⊃ uniform topology ⊃ Product topology.

let $x = (x_\alpha)_{\alpha \in J}$

and let $U = \prod U_\alpha$ be a basis element about x in the product topology.

Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be the indices for which $\boxed{U_\alpha \neq R}$ then for each i, choose $\epsilon_i > 0$, so that ϵ_i-ball centred at x_{α_i} in the $\bar{d}$ metric is contained in U_{α_i}, $\because$ each U_{α_i} is open in $\mathbb{R}$.

i.e., $B_{\bar{d}}(x_{\alpha_i}, \epsilon_{\alpha_i}) \subset U_{\alpha_i}$

Let $\epsilon = \min\{\epsilon_1, \ldots \epsilon_n\}$.

CLAIM:

$B_{\bar{\rho}}(x, \epsilon) \subset U$

Let $y \in B_{\bar{\rho}}(x, \epsilon)$

$\Rightarrow \bar{\rho}(x, y) < \epsilon$

$\Rightarrow \sup\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in J\} < \epsilon$

$\Rightarrow \bar{d}(x_\alpha, y_\alpha) < \epsilon, \forall \alpha \in J.$

$\Rightarrow y_\alpha \in B_{\bar{d}}(x_\alpha, \epsilon) \; \forall \alpha.$

In particular,

$y_{\alpha_i} \in B_{\bar{d}}(x_{\alpha_i}, \epsilon) \subset B_{\bar{d}}(x_{\alpha_i}, \epsilon_{\alpha_i}) \subset U_{\alpha_i}$, $i = 1, 2 \ldots n$

i.e., $y_{\alpha_i} \in U_{\alpha_i}$, $i = 1, 2 \ldots n$.

$y_\alpha \in \mathbb{R}$, $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$.

$\therefore y = (y_\alpha)_{\alpha \in J} \in U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n}$

$\therefore y \in \prod U_\alpha$.

i.e., $y \in U$.

$\Rightarrow B_{\bar{\rho}}(x, \epsilon) \subset U$.

$\therefore$ Uniform topology $\supset$ Product topology.

T.P: Box topology $\supset$ Uniform topology.

Let $x = (x_\alpha)_{\alpha \in J}$.

Let $\epsilon > 0$ be given, then $B_{\bar{\rho}}(x, \epsilon)$ is a basis element containing x in the uniform topology.

Consider, $U = \prod_{\alpha \in J} (x_\alpha - \epsilon/2, x_\alpha + \epsilon/2)$ then

U is a basis element containing x in the Box topology on $\mathbb{R}^J$.

CLAIM :-

$U \subset B_{\bar{\rho}}(x, \epsilon)$.

Let $y \in U$.

$\therefore y_\alpha \in (x_\alpha - \epsilon/2, x_\alpha + \epsilon/2), \forall \alpha \in J$.

$\therefore |x_\alpha - y_\alpha| < \epsilon/2, \forall \alpha \in J$.

$\sup \{ |x_\alpha - y_\alpha| / \alpha \in J \} < \epsilon/2 < \epsilon < 1$.

$\therefore \sup \{ \bar{d}(x_\alpha, y_\alpha) / \alpha \in J \} < \epsilon$.

i.e. $\bar{\rho}(x, y) < \epsilon$.

$\Rightarrow y \in B_{\bar{\rho}}(x, \epsilon)$

$\therefore U \subset B_{\bar{\rho}}(x, \epsilon)$.

$\therefore$ Box topology $\supset$ Uniform topology.

Hence Box topology $\supset$ uniform topology $\supset$ Product topology.

Thm 20.5

Let $\bar{d}(a, b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define $D(x, y) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$.

Then D is a metric that induces the product topology on $\mathbb{R}^{\omega}$.

Proof:

1st we shall prove that D is a metric on $\mathbb{R}^{\omega}$.

i). $D(x,y) \geq 0$, $\because$ $\bar{d}(x_i, y_i) \geq 0 \;\forall\; i$ for $\bar{d}$ is a metric and $D(x,y) = 0$.

$$\iff \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \;/\; i = 1, 2, \dots \infty \right\} = 0$$

$$\iff \frac{\bar{d}(x_i, y_i)}{i} = 0, \quad i = 1, 2 \dots \infty$$

$$\iff \bar{d}(x_i, y_i) = 0, \quad i = 1, 2 \dots \infty$$

$$\iff x_i = y_i, \quad i = 1, 2 \dots \infty$$

$$\iff (x_i)_{i=1}^{\infty} = (y_i)_{i=1}^{\infty}$$

$$\iff x = y.$$

ii). $D(x,y) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \;/\; i = 1, 2 \dots \infty \right\}$.

$$= \sup \left\{ \frac{\bar{d}(y_i, x_i)}{i} \;/\; i = 1, 2 \dots \infty \right\}$$

$$= D(y, x).$$

iii). Triangular inequality.

Let $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty}$, $z = (z_i)_{i=1}^{\infty}$.

W.K.T

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i), \quad i = 1, 2 \dots \infty$$

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}, \quad i = 1, 2 \dots \infty$$

$$\sup \left\{ \frac{\bar{d}(x_i, z_i)}{i} \;/\; i = 1, 2 \dots \infty \right\} \leq \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \;/\; i = 1, 2 \dots \infty \right\} + \sup \left\{ \frac{\bar{d}(y_i, z_i)}{i} \;/\; i = 1, 2 \dots \infty \right\}$$

i.e; $D(x,z) \leq D(x,y) + D(y,z)$ region

$\therefore$ D is a metric on $\mathbb{R}^\omega$.

Next we shall prove that D topology and product topology on $\mathbb{R}^\omega$ are same.

1st we have to prove that P.T $\supset$ D.T

Let U be open in the metric topology (D) and let $x \in U$ we can find an open set V in the product topology such that $x \in V \subset U$.

Choose an ϵ-ball $B_D(x, \epsilon)$ lying in U.

Then choose N, Large enough that $\frac{1}{N} < \epsilon$.

Let V be the basis element for the product topology.

i.e; $V = (x_1 - \epsilon, x_1 + \epsilon) \times \dots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \dots$

CLAIM:

$V \subset B_D(x, \epsilon)$:

Let $y = (y_i)_{i=1}^{N} \in V$.

$\therefore y_i = (x_i - \epsilon, x_i + \epsilon)$ for $i = 1, 2, \dots N$

and $y_i \in \mathbb{R}$ if $i \neq 1, 2, \dots N$.

$\therefore |x_i - y_i| < \epsilon \quad \forall \; i = 1, 2 .. N$

i.e; $|x_i - y_i| < \epsilon \;\forall\; i \leq N$

$$\Rightarrow \frac{|x_i - y_i|}{i} < \frac{\epsilon}{i}$$

$$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\epsilon}{i} < \epsilon \;\forall\; i \leq N \longrightarrow (1)$$

$\forall\ i > N, \frac{i}{N} > 1 \longrightarrow$ (2)

But $\frac{1}{N} < \epsilon$

$\Rightarrow \frac{i}{N} < i\epsilon \longrightarrow$ (3) we get,

Combine (2) & (3)

$1 < \frac{i}{N} < i\epsilon \longrightarrow$ (4)

But $\bar{d}(x_i, y_i) \leq 1 \longrightarrow$ (5)

Combining (4) & (5)

$\bar{d}(x_i, y_i) < i\epsilon$

$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} < \epsilon \ \forall\ i > N \longrightarrow$ (6)

From (1) & (6)

$\frac{\bar{d}(x_i, y_i)}{i} < \epsilon \ \forall\ i$

$\sup \left\{ \frac{\bar{d}(x_i, y_i)}{i}, \ i = 1, 2 \ldots \infty \right\} < \epsilon$

i.e., $D(x, y) < \epsilon$

$\therefore y \in B_D(x, \epsilon)$

$\therefore V \subset B_D(x, \epsilon)$.

i.e., Product topology $\supset$ D topology on $\mathbb{R}^\omega$. $\longrightarrow$ (7)

T.P: D-topology on $\mathbb{R}^\omega \supset$ Product topology.

Let $x = (x_i)_{i=1}^{\infty} \in \mathbb{R}^\omega$.

Let $U = \prod U_i$, where $U_i = U_i$ for $i = 1, 2 \ldots n$

$= \mathbb{R}$ for $i \neq 1, 2 \ldots n$.

Let U be the basis element in the product topology containing x.

$\therefore x_i \in U_i$ for $i = 1, 2 \ldots n$ where each

U_i is open in $\mathbb{R}$. Then $\exists\ \epsilon > 0$ such that

$x_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$.

$\therefore (x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2) \times \ldots$

$\times (x_n - \epsilon_n, x_n + \epsilon_n)$.

$R \times R \times \ldots \subset U$.

Without loss of generality choose $\epsilon_i < 1$ and let

$\epsilon = \min \left\{ \frac{\epsilon_i}{i} \,/\, i = 1, 2 \ldots n \right\}$

CLAIM:

$B_D(x, \epsilon) \subset U$.

Let $y = (y_i)_{i=1}^{\infty} \in B_D(x, \epsilon)$

$\therefore D(x, y) < \epsilon$.

$\therefore \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \,/\, i = 1, 2 \ldots \infty \right\} < \epsilon$.

$\therefore \frac{\bar{d}(x_i, y_i)}{i} < \epsilon \leq \frac{\epsilon_i}{i}$ for $i = 1, 2 \ldots \infty$

$\therefore \bar{d}(x_i, y_i) < \epsilon_i$ for $i = 1, \ldots \infty$

$\min \{ |x_i - y_i|, 1 \} < \epsilon_i$, for $i = 1, \ldots \infty$.

$\Rightarrow |x_i - y_i| < \epsilon_i$ for $i = 1, \ldots \infty$

$\Rightarrow y_i \in (x_i - \epsilon_i, x_i + \epsilon_i)$ for $i = 1, \ldots \infty$

$\Rightarrow y \in (x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2)$

$\times \ldots \times (x_n - \epsilon_n, x_n + \epsilon_n) \times R \times R \times \ldots$

$\Rightarrow y \in U$

$\therefore B_D(x, \epsilon) \subset U$.

$\Rightarrow$ D- topology $\supset$ product topology.

From (7) & (8)

The D- topology Co- inside with product topology in $\mathbb{R}^{\omega}$.

$\therefore$ The Product topology on $\mathbb{R}^{\omega}$ is induced by the metric D.

Hence proved.

Sec. 21. The Metric Topology (cont).

Thm 21.1.

U.Q (8) 5M Let $f: X \to Y$; let X and Y be metrizable with metrices d_X and d_Y respectively. Then Continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

Proof:

Suppose that f is continuous.

Gn: $x \in X$ and $\epsilon > 0$

Consider any element $B_{d_Y}(f(x), \epsilon)$ in Y which is open.

$\therefore$ $f^{-1}(B_{d_Y}(f(x), \epsilon))$ is open in X and contains the point x. ($\because$ f is continuous).

$\therefore$ $\exists$ some δ- ball.

$$B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon)).$$

Let $y \in B_{d_X}(x, \delta)$.

$\Rightarrow y \in f^{-1}(B_{d_Y}(f(x), \epsilon))$

$\Rightarrow f(y) \in B_{d_Y}(f(x), \epsilon)$

$\Rightarrow d_Y(f(x), f(y)) < \epsilon$

i.e., $y \in B_{d_X}(x, \delta) \Rightarrow f(y) \in B_{d_Y}(f(x), \epsilon)$

$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Conversely,

Suppose that the $\epsilon - \delta$ condition is satisfied.

T.P: $f : X \to Y$ is continuous.

Let V be open in Y.

CLAIM: $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$

$\therefore f(x) \in V$ there is an ϵ-ball

$B_{d_Y}[f(x), \epsilon) \subset V$. By $\epsilon - \delta$ condition there is a δ-ball $B_{d_X}(x, \delta)$ centered at x such that,

$f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon) \subset V$.

i.e., $f(B_{d_X}(x, \delta)) \subset V$.

$B_{d_X}(x, \delta) \subset f^{-1}(V)$

$\therefore f^{-1}(V)$ is open in X.

$\therefore f : X \to Y$ is continuous.

H.P//.

Lemma 21.2 (The Sequence Lemma):

R ⊛ U.Q. 5M

Let X be a topological space, let $A \subset X$. If there is a sequence (set of value) of points of A converging to x, then $x \in \bar{A}$ the converse holds if X is metrizable.

Thm 17.5

Proof:

Suppose that $x_n \to x$, where $x_n \in A$.

Then every neighbourhood U of x contains a pt of A.

By thm "Let A be a subset of the topological Space X. Then $x \in \bar{A}$ iff every open set U containing x intersects."

$\therefore x \in \bar{A}$

Conversely,

Suppose that X is metrizable and $x \in \bar{A}$.

Let d be a metric for the topology of x.

For each the integer n, take the neighbourhood $B_d(x, 1/n)$ of radius $1/n$ of x and choose x_n to be a point of its intersection with A.

Claim: $x_n \to x$.

Let U be an open set containing x.

$\therefore \exists$ some ϵ - ball $B_d(x, \epsilon)$ such that $B_d(x, \epsilon) \subset U$.

If we choose N so that $\frac{1}{N} < \epsilon$ then

$x_i \in B_d(x, \frac{1}{N}), \forall i \geq N$

$d(x, x_i) < \frac{1}{N}, \forall i \geq N$

$\Rightarrow d(x, x_i) < \frac{1}{N} \epsilon, \forall i \geq N$.

$\Rightarrow d(x, x_i) < \epsilon, \forall i \geq N$.

$\Rightarrow x_i \in B_d(x, \epsilon), \forall i \geq N$

i.e. $x_i \in U, \forall i \geq N$

$\therefore x_n \to x$.

Hence proved

U.Q ⊛ 5M

Thm 21.3

Let $f: X \to Y$. If the function 'f' is continuous, then every convergent sequence $x_n \to x$ in X. The seq of $f(x_n)$ cgs to $f(x)$. The converse holds if X is metrizable.

Proof:

Assume that f is continuous.

Gn: $x_n \to x$

T.P: $f(x_n) \to f(x)$

Let V be a nbd of $f(x)$.

Then $f^{-1}(V)$ is a nbd of x. ($\because f$ is continuous)

$\because x_n \to x$, there is an N such that,

$x_n \in f^{-1}(V)$ for $n \geq N$

$\Rightarrow f(x_n) \in V$ for $n \geq N$

$\therefore f(x_n) \to f(x)$.

Conversely,

Assume that the cgt seq condition is satisfied.

T.P: $f: X \to Y$ is continuous.

Let X be metrizable and let A be a subset of X.

Claim:

$f(\bar{A}) \subset \overline{f(A)}$

Let $f(x) \in f(\bar{A}) \Rightarrow x \in \bar{A}$

~~xx~~ by f^{-1}

$\therefore$ by Sequence Lemma.

There is a sequence x_n of points of A converging to x.

By assumption, the seq $f(x_n) \to f(x)$

$\because$ $f(x_n) \in f(A)$, by the seq. lemma,

that $f(x) \in \overline{f(A)}$

$\therefore$ $f(\overline{A}) \subset \overline{f(A)}$

$\Rightarrow$ f is continuous.

H.P.

Lemma 21.4.

The addition, subtraction and multiplication operations are continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation (Division) is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

Proof.

Let the metrices d and ρ are defined by

$d(a,b) = |a-b|$ on $\mathbb{R}$ and

$\rho((x,y),(x_0,y_0)) = \max\{|x-x_0|, |y-y_0|\}$ on $\mathbb{R}^2$

T.P: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(+) Let $(x,y) \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$ and take $\delta = \epsilon/2$

Let $\rho((x,y),(x_0,y_0)) < \delta = \epsilon/2$

$\Rightarrow \max\{|x-x_0|, |y-y_0|\} < \epsilon/2$

$\Rightarrow |x-x_0| < \epsilon/2$ (or) $|y-y_0| < \epsilon/2$

$\therefore |x-x_0| + |y-y_0| < \epsilon/2 + \epsilon/2 = \epsilon.$

$\Rightarrow d(x+y, x_0+y_0) < \epsilon$

$\because d(x+y, x_0+y_0) = |(x+y) - (x_0+y_0)|$

$= |x-x_0+y-y_0|$

$\leq |x-x_0| + |y-y_0|$

$\therefore \rho((x,y),(x_0,y_0)) < \delta \Rightarrow d(x+y, x_0, y_0) < \epsilon$

$\therefore$ By $\delta - \epsilon$ condition.

(+) $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(−) $|||^{ly}$ we can prove : $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

for (×) Let $3\delta = \min\left\{\frac{\epsilon}{|x_0|+|y_0|+1}, \sqrt{\epsilon}\right\}$, $0<\epsilon<1$.

Let $\rho((x,y),(x_0,y_0)) < 3\delta$

$\Rightarrow \max\{|x-x_0|, |y-y_0|\} < 3\delta.$

$\Rightarrow |x-x_0| < 3\delta$ or $|y-y_0| < 3\delta$

$\Rightarrow |x-x_0| < \min\left\{\frac{\epsilon}{|x_0|+|y_0|+1}, \sqrt{\epsilon}\right\}$ (or)

$|y-y_0| < \min\left\{\frac{\epsilon}{|x_0|+|y_0|+1}, \sqrt{\epsilon}\right\}.$

$\Rightarrow |x-x_0| < \frac{\epsilon}{|x_0|+|y_0|+1}$, $|y-y_0| < \frac{\epsilon}{|x_0|+|y_0|+1}$

(or)

$|x-x_0| < \sqrt{\epsilon}$; $|y-y_0| < \sqrt{\epsilon}.$

$\therefore |x-x_0| < \frac{\epsilon}{|y_0|}$ (or) $|x-x_0| < \sqrt{\epsilon}.$

and $|y-y_0| < \frac{\epsilon}{|x_0|}$ (or) $|y-y_0| < \sqrt{\epsilon}.$

Let $|x_0||y-y_0| + |y_0||x-x_0| + |x-x_0||y-y_0|$

$< |x_0| \cdot \frac{\epsilon}{|x_0|} + |y_0| \frac{\epsilon}{|y_0|} + \sqrt{\epsilon} \cdot \sqrt{\epsilon}$

$< \epsilon + \epsilon + \epsilon$

$< 3\epsilon.$

$\therefore d(xy, x_0 y_0) = |xy - x_0 y_0|$

$= |xy - x_0 y - x y_0 + x_0 y_0 + x y_0 + x_0 y - x_0 y_0 - x_0 y_0|$

$= |(x - x_0)(y - y_0) + x_0(y - y_0) + y_0(x - x_0)|$

$\leq |x - x_0||y - y_0| + |x_0||y - y_0| + |y_0||x - x_0|$

$\leq 3\epsilon.$

$\therefore \rho((x, y), (x_0, y_0)) < 3\delta \Rightarrow d(xy, x_0 y_0) < 3\epsilon.$

$\therefore f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

III^{ly} we can prove, $(\div) : \mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$ is continuous.

H.P.//

U.Q (x) 5M.

Thm 21.5

If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous function, then $f+g$, $f-g$ and $f \cdot g$ are continuous. If $g(x) \neq 0 \ \forall x$, then f/g is continuous.

Proof:

The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by,

$h(x) = f(x) \times g(x)$ is continuous.

By pasting lemma,

The function $f+g$ equals the composite of h and the addition operation $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\because$ 'h' and 'f' are continuous.

$\Rightarrow$ h composite addition operation is continuous

$\Rightarrow f+g$ is continuous.

111[4]

arguments apply to $f+g$, $f\cdot g$ and f/g.

Def: uniformly Converges:

Let $f_n: X \to Y$ be a sequence of function from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence $\{f_n\}$ cgs uniformly to the fun $f: X \to Y$ if given $\epsilon > 0$, $\exists$ an integer N such that

$d(f_n(x), f(x)) < \epsilon$ $\forall n > N$ and all x in X.

Thm 21.6 (uniform limit Thm):

Let $f_n: X \to Y$ be a seq of continuous function from the topological space X to the metric space Y. If (f_n) cgs uniformly to f, then f is continuous

Proof:

Gn: $f_n: X \to Y$ be continuous and (f_n) cgs uniformly to f.

T.P: $\therefore$ f is continuous.

Let V be open in Y.

Let x_0 be a point of $f^{-1}(V)$.

We have to find a nbd U of x_0 such that $f(U) \subset V$.

Let $y_0 = f(x_0)$

Choose ϵ, so that the ϵ-ball $B(y_0, \epsilon) \subset V$.

Then using (f_n) cgs uniformly to f, choose N

so that $d(f_n(x), f(x)) < \epsilon/3 \rightarrow$ ①

$\because f_N$ is continuous, choose a ndb. U of x_0 $\ni$

$f_N(U) \in B_d(f_N(x_0, \epsilon/3)) \;\forall\; n \geq N.$

Let $x \in U$ then $f_N(x) \in f_N$

$\therefore f_N(x) \subset B_d(f_N(x_0), \epsilon/3)$

$\rightarrow d(f_N(x), f_N(x_0)) < \epsilon/3 \rightarrow$ ②

CLAIM: $f(U) \subset B_d(y_0, \epsilon) \subset V.$

If $x \in V$ then

$d(f(x), f_N(x)) < \epsilon/3$ (by choice of N)

$d(f_N(x), f_N(x_0)) < \epsilon/3$ (choice of U)

$d(f_N(x_0), f(x_0)) < \epsilon/3$ (" of N)

$\therefore d(f(x), f(x_0)) = |f(x) - f(x_0)|$

$= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)|$

$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

$d(f(x), f(x_0)) < \epsilon$

$\Rightarrow f(x) \in B_d(f(x_0), \epsilon)$

$\Rightarrow f(V) \in B_d(y_0, \epsilon) \subset V.$

$f(U) \subset V$

$\Rightarrow U \in f^{-1}(V)$

$\therefore f^{-1}(V)$ is open in X.

$\therefore f: X \rightarrow Y$ is continuous.

Additional Resource :

http://mathforum.org

http://ocw.mit.edu/ocwweb/Mathematics

http://www.opensource.org

http://en.wikipedia.org

Practice Questions:

Question Bank

Section – A

1.Define Continuous function

2.Define homeomorphism

3.Define topological imbedding

4.State Pasting Lemma.

5.Define J Tuple.

6.Define product space.

7.Define Matric topology.

8.Define metric space.

9.State Sequence lemma

10.Define Converges uniformly.

11.Define Uniform limit Theorem.

Section – B

1.State and prove Pasting Lemma

2.Let $f: X \to Y$; let X and Y be metrizable with metrics d_x and d_y respectively, then prove that the Continuity of f is equivalent to the requirement that given $x \in X$ and given $\in > 0$,there exits $\delta > 0$ Such that $d_x(x, y) \Longrightarrow d_y(f(x), f(y)) < \in$

3.The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser set the box topology; There three topologies are all different if $\mathcal{J}$ is infinite.

4.Let $\{X_\alpha\}$ ba an indexed family of space;let $A_\alpha \subset X_\alpha$ for each α .If πX_α is given either the product on the box topology, Prove that $\pi \bar{A}_\alpha = \overline{\overline{\pi A_\alpha}}$

5.State and prove the Sequence lemma.

6.If X is a topological space and if $f, g: X \to \mathbb{R}$ are continuous functions, prove that $f + g, f - g$ and $f.g$ are continuous and if $g(x) \neq 0$ for all x . Prove that f/g is continuous

7.Define box topology and product topology. Explain how does the product topology differ from The box topology.

8.Let X and Y be topological spaces. Prove that the map f is continuous if X can be written as the union of open set U_α such that f/U_α is continuous for each α

9.Let $f_\alpha: X \to Y$ be a sequence of continuous functions from the topological space X to the metric Space Y. If f_n converges uniformly to f , prove that f is continuous.

9. Let $f: A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$, Prove that f is continuous if and only if the functions $f_1: A \to X$ and $f_2: A \to Y$ are continuous.

Section – C

1.Let X and Y be topological spaces , let $f: X \to Y$, Then prove that the following are equivalent

(a) f is continuous

(b) For every subset A of X .one has $f(\overline{A}) \subset \overline{f(A)}$

(c) For every closed set B of Y ,the set $f^{-1}(B)$ is closed in X

(d)For each $x \in X$ and neighbourhood V of $f(x)$ there is a neighbor hood U of x such that $f(U) \subset V$

2. Prove that The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the Product topology on $\mathbb{R}^n$

3.State and prove pasting lemma.

4.Let $f: A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$,Prove that f is continuous if and only if the coordinate functions of f are continuous.

5.Let X, Y $be\ two\ topological\ spaces, P.T\ f: X \to Y$ is continuous,if and only if the inverse image of Every closed set is closed.

6.Let $\bar{d}(a,b) = \min\{|a-b|,1\}$ be the standard bounded metric on $\mathbb{R}$,if $\bar{x}\ and\ \bar{y}$ are two points of $\mathbb{R}^w$ Define $(\bar{x},\bar{y}$) = sup $\{\frac{\overline{d}(x_i, y_i)}{i}\}$ P.T D is a metric that induces the product topology on $\mathbb{R}^w$

Recommended Text : James R. Munkres, Topology (2nd Edition) Pearson Education Pve. Ltd., Delhi-2002 (Third Indian Reprint)