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II M.Sc Mathematics – Semester - III

E-Notes (Study Material)

Core Course : Topology **Code: 23PMA33**
UNIT-I : Topological spaces : Topological spaces – Basis for a topology – The order topology – The product topology on X × Y – The subspace topology – Closed sets and limit points.
Learning Objectives: To study Topological spaces , Basis for a topology, The order topology , The product topology on X × Y , The subspace topology , Closed sets and limit points.
Course Outcome: Define and illustrate the concept of topological spaces and the basic definitions of open sets, neighbourhood, interior, exterior, closure and their axioms for defining topological space.

Overview:

Topological Spaces:
A topological space is a set with a structure that tells us which subsets are "open." This helps define concepts like continuity and limits.

Basis for a Topology:
A basis is a set of "building blocks" that we use to create a topology. The open sets in the topology are unions of these basis sets.

Order Topology:
The order topology is a way to make a totally ordered set (like the real numbers) into a topological space, using intervals based on the order of the elements.

Product Topology:
The product topology is used to create a topology on the product of two spaces, like X×YX \times YX×Y, by combining open sets from each space.

Subspace Topology:
The subspace topology is the topology on a subset of a space, created by looking at the open sets of the larger space and intersecting them with the subset.

Closed Sets and Limit Points:

A closed set is one where its complement is open. A limit point is a point where every neighborhood around it contains points from a given set (other than the point itself).

This unit gives the basic tools to work with and understand topological spaces.

UNIT - 1

TOPOLOGY

A Topology on a set X is a collection $\mathcal{J}$ of subset X of subset having the following property.

i) ϕ and X are in $\mathcal{J}$.

ii) The union of element of any finite sub collection of $\mathcal{J}$ is in $\mathcal{J}$.

iii) The intersection of the element of any finite Subcollection of $\mathcal{J}$ is in $\mathcal{J}$.

Topological Spaces:

A set X together with a topology $\mathcal{J}$ defined on it is called a topological spaces denoted by $(X, \mathcal{J})$

Open set:

If X is a topological spaces with topology $\mathcal{J}$ we say that a subset U of X is an open set of X if U belongs to the collection $\mathcal{J}$.

Note:

A topological space is a set X together a collection of subsets of X called open sets such that ϕ and X are both open and such that orbitary union and finite intersection of open sets are open.

Eg:

Let X be a three element set, $X = \{a, b, c\}$ then X has namely different topologies but not every collection of subset of X is a topology on X.

Soln:

Let $X = \{a, b, c\}$

$\mathcal{J}_1 = \{X, \phi, \{a\}\}$

$\mathcal{J}_2 = \{X, \phi, \{a\}, \{b\}\}$

$\mathcal{J}_3 = \{X, \phi, \{a\}, \{b\}, \{c\}\}$

$\mathcal{J}_4 = \{X, \phi\}$

$\mathcal{J}_5 = \{X, \phi, \{a\}, \{ab\}\}$

$\mathcal{J}_6 = \{X, \phi, \{a, b\}, \{b, c\}\}$

$\mathcal{J}_7 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$

$\mathcal{J}_8 = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$

$\mathcal{J}_9 = \{X, \phi, \{a, b\}, \{b, c\}, \{c, a\}\}$

$\mathcal{J}_{10} = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$

Here, $\mathcal{J}_1, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_7, \mathcal{J}_8, \mathcal{J}_{10}$ are all topologes on X

But $\mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_6, \mathcal{J}_9$ are not topologes on X under (ii)

III^{ly}

$\mathcal{J}_6, \mathcal{J}_9$ are not topologes on X under (iii)

$\therefore$ We conclude that every collection of subsets of X is not a topology on X.

Ex: 1 Discrete Topology:

If X is any set, collection of all subsets of X is a topology on X, it is called the discrete Topology.

Ex: 2 Indiscrete Topology:

If the collection of consisting of X and ϕ only is called indiscrete Topology. (or) the trivial topology.

Ex: 3 Let: Finite Complement Topology: (Thm): 13.1.

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Let X be a set. Let $\mathcal{J}_f$ be the collection of all subset U of X such that $X - U$ either is finite or is all of X then $\mathcal{J}_f$ is a topology on X called the finite Complement Topology.

Proof:

Sec i) Since $X - X$ is finite and $X - \phi$ is all of X

$\therefore$ Both X and ϕ are in $\mathcal{J}_f$.

ii) If $\{U_\alpha\}$ is an indexed family of non empty elements $\mathcal{J}_f$, to show that $\cup U_\alpha$ is in $\mathcal{J}_f$

By de-morgan's Law

$$X - \cup U_\alpha = \cap (X - U_\alpha) \longrightarrow (1)$$

$\because$ each set $X - U_\alpha$ is finite or X.

$\Rightarrow \cap (X - U_\alpha)$ is finite (or) X.

$\Rightarrow X - \cup U_\alpha$ is finite (or) X ($\because$ by (1))

$\Rightarrow \cup U_\alpha$ is in $\mathcal{J}_f$

iii) To show $\cap U_\alpha$ is in $\mathcal{J}_f$

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_P) \rightarrow ②$$

$\because$ each set $X - U_i$ is finite then

$\bigcup_{i=1}^{n} (X - U_i)$ is finite.

$X - \bigcap_{i=1}^{n} U_i$ is finite [$\because$ by (2)]

$\therefore \cap U_i$ is in $\mathcal{J}_f$.

$\therefore \mathcal{J}_f$ is a topology on X.

Ex: ~~4~~. Let X be a set. Let $\mathcal{J}_c$ be the collection of all subsets U of X such that $X - U$ either is countable (or) is all of X. Then $\mathcal{J}_c$ is topology on X.

Small
~~Finer~~ topology:

~~In~~ Suppose that $\mathcal{J}$ & $\mathcal{J}'$ are two topology on a given set X. If $\mathcal{J}' \supseteq \mathcal{J}$, We say that $\mathcal{J}'$ is finer than $\mathcal{J}$. If $\mathcal{J}'$ properly contains $\mathcal{J}$ we say that $\mathcal{J}'$ is strictly finer than $\mathcal{J}$, we also say that $\mathcal{J}$ is coarsser than $\mathcal{J}'$ (or) strictly coarser in these two respective situations. We say $\mathcal{J}$ is comparable $\mathcal{J}'$ if either $\mathcal{J}'$ contains $\mathcal{J}^*$ (or) $\mathcal{J} \supset \mathcal{J}'$.

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Basis for Topology :

If X is a said a basis for a topology on X is a collection $\mathcal{B}$ of a subset of X (called basis element) such that

i) For each $x \in X$ there is atleast one basis element $\mathcal{B}$ containing x.

ii) If x belongs to the intersection two basis B_1 and B_2, then there is a basis element B_3 containing x.

$$B_3 \subset B_1 \cap B_2.$$

If $\mathcal{B}$ satisfies these two conditions then we defined a topology $\mathcal{B}$ generated by $\mathcal{B}$ as follows.

A subset U of X is said a open in X (i.e. To be an element of $\mathcal{T}$) if for each $x \in U$, there is basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Note:

Each basis element is itself on element of $\mathcal{T}$.

Ex:

Let $\mathcal{B}$ be the collection of all circular regions (interior of circle) in the plane then $\mathcal{B}$ satisfies both conditions for a basis.

In the topology generated by $\mathcal{B}$ a subset U of the plane is open if every x in U lies in some circular region contained in U.

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Lemma : (*)

Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all union of element of $\mathcal{B}$.

Proof :

A collection of element $\mathcal{B}$, they are also a element of $\mathcal{T}$ because $\mathcal{T}$ is a topology, there union is in $\mathcal{T}$.

Conversely,

Gn : $U \in \mathcal{T}$, choose for each $x \in U$ and element B_x of $\mathcal{B}$ such that $x \in B \in U$.

Then $U = \bigcup_{x \in U} B_x$

So U equals an union elements of $\mathcal{B}$.

Hence $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Hence the proof.

Lemma : 13.2

Let X be a topological spaces. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set $U \notin x$ and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$ then $\mathcal{C}$ is a basis for the topology of X.

Proof !

TP : $\mathcal{C}$ is a basis of some topology on X.

Gn: $x \in X$, since X is itself an open set.

There is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$.

To verify the 2nd conditions

Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$

$\because C_1, C_2$ are open then $C_1 \cap C_2$ is an open subsets of X.

$\therefore$ By hypothesis we can find $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

$\therefore \mathcal{C}$ is a basis for some basis topology T' on X.

Let $\mathcal{C}^{J}$ is a script topology on X.

~~the topology~~ Now we must such that the topology $\mathcal{J}'$ generated by $\mathcal{C}$ equals to the topology.

If $U \in \mathcal{J}$ and if $x \in U$ then by hypothesis, an element C of $\mathcal{C}$ such that $x \in \mathcal{C} \subset U$.

$\therefore$ By def, $U \in \mathcal{J}'$.

i.e., $\mathcal{J} \subset \mathcal{J}' \longrightarrow (1)$

Conversely

If $W \subset \mathcal{J}'$, by Lemma (13.1)

then W equals a union of element of $\mathcal{C}$.

$\because$ each element $\mathcal{C} \in \mathcal{J}$ and $\mathcal{J}$ is a Topology,

W is also belongs to $\mathcal{J}$.

i.e. $\mathcal{T}' \subset \mathcal{T} \longrightarrow$ (2)

From (1) & (2)

$$\mathcal{T} = \mathcal{T}'$$

Hence $\mathcal{C}$ is a basis for the topology of X.

Hence the proof.

Lemma - 13 : 3 :

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the Topology $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following are equivalents

i) $\mathcal{T}'$ is finite then $\mathcal{T}$

ii) For each $x \in X$ and basis element $B \in \mathcal{B}$ Containing X, there is basis element $B \in \mathcal{B}$ such that $x \in B' \subset B$

Proof:

(ii) $\Rightarrow$ (i)

T.P: $\mathcal{T}' \supset \mathcal{T}$ is finite then $\mathcal{T}$.

i.e. $\mathcal{T}' \supset \mathcal{T}$.

If $U \in \mathcal{T}$ We have to prove that $U \in \mathcal{T}'$.

Let $x \in U$

$\because$ $\mathcal{B}$ generated $\mathcal{T}$ there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

By (ii), there exist an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$ then $x \in B' \subset U$.

But $B' \in \mathcal{B}'$ is a basis for $\mathcal{T}'$.

$\therefore U \in \mathcal{T}'$

$\Rightarrow \mathcal{T}' \supset \mathcal{T}$

We now prove (i) => (ii)

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for topology $\mathcal{T}$ and $\mathcal{T}'$,

Let $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

Now $B \in \mathcal{B}$ is a basis for $\mathcal{T}$.

$\therefore B \in \mathcal{T}$

By (i) $\mathcal{T}' \supset \mathcal{T}$

$\Rightarrow \mathcal{T} \in \mathcal{T}'$

$\because \mathcal{T}'$ is generated by B'

There is an $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Hence the proof.

2M U.Q Standard Topology:

If $\mathcal{B}$ is the collection of all open intervals in the real line $(a, b) = \{x / a < x < b\}$, the topology generated by $\mathcal{B}$ is called the Standard topology on the real line.

Lower limit Topology:

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b) = \{x / a \leq x < b\}$, where ~~α~~ is $a < B$, then the topology generated by $\mathcal{B}'$ is called the Lower Limit topology on $\mathbb{R}$. When $\mathbb{R}$ is given the Lower limit topology it denote by $\mathbb{R}_l$.

K Topology :

Let K denote the set of all numbers of the form $\frac{1}{n}$ for $n \in Z_+$ and Let $\mathcal{B}''$ be the collection of all open intervals (a, b) along with all set of the form $(a, b) - K$.

The topology generated by $\mathcal{B}''$ will be Called the k - Topology on $\mathbb{R}$, when $\mathbb{R}$ is given then its denoted by $\mathbb{R}_K$.

Lemma : 13.4.

The topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are strictly finer then the standard topology on $\mathbb{R}$ but aren't comparable with 1 another.

Proof :

Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_k$ are respectively.

Gn : (a, b) be a basis element for $\mathcal{T}$ and let a point $x \in (a, b)$.

Let $[x, b)$ be a basis element for $\mathcal{T}'$ and it contains x and Lies in (a, b).

On the other hand, if $[x, d)$ be basis element for , there is no interval $(a. b)$ that contains x and Lies in $[x, d]$, $\mathcal{T}' \supset \mathcal{T}$.

Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Now, given a basis element (a,b) for $\mathcal{T}$ and a point x of (a,b), this same is a basis element for $\mathcal{T}''$, that contains X.

On the other hand,

Gn: The basis element $\mathcal{B} = (-1,+1) - K$ for $\mathcal{T}''$ and the point $\underset{\text{zero}}{0}$ of $\mathcal{B}$.

There is no open interval that contains zero and the lies in $\mathcal{T}$.

$\therefore \mathcal{T}'' \supset \mathcal{T}$

Thus $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

Hence the topology $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ and also they are not compeurable with 1 another.

Hence the proof.

Subbasis

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose unions equals X. The topology generated by a subbasis $\mathcal{S}$ is defined to the collection $\mathcal{T}$ of all union of finite intersection of element of $\mathcal{S}$ that is

$\cup(\cap S_i) \in \mathcal{T}$.

The order of element : Order Topology :

Let S be a set with a simple order relation

Assume X as more than one element.

Let $\mathcal{B}$ be the collection of all sets the followings types

i) All open intervals (a,b) in X

ii) All integral of the form $[a_0, b)$.

Let the First order partial derivative

Where a_0 is the smallest element (if any) of X.

iii) All intervals of the form $(a, b_0]$

Where b_0 is the Largest element (if any) of X.

The Collection $\mathcal{B}$ is a basis for a topology on X which is called the ordered topology.

If X has no smallest element, there are no sets of type 2 and if X has no Largest element, there are no sets of type 3.

Eg:

The standard topology of $\mathbb{R}$ is just order topology derived from the usual order on $\mathbb{R}$.

Open rays and closed rays:

If X is an ordered set and 'a' is element of X, there are 4 subsets of X that are called the rays determined by 'a'.

They have the following

i) $(a, +\infty) = \{x \mid x > a\}$

ii) $(-\infty, a) = \{x \mid x < a\}$

iii) $[a, +\infty] = \{x \mid x \geq a\}$

iv) $(-\infty, a] = \{x \mid x \leq a\}$

Sets of the first two types are called Open rays. And the set of the Last two types are called closed rays.

The product Topology on X into Y:

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Product Topology:

Let X and Y be topological spaces, the product topology on X into Y is the topology having collection of $\mathcal{B}$ of all sets of form $U \times V$ where U is open subset of X and where V is Open sub set of Y.

U.Q 5M

Thm: 15.1

If $\mathcal{B}$ is a basis for a topology of X and $\mathcal{C}$ is a basis for the topology of Y then the Collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is basis for the topology of $X \times Y$.

Proof:

Let W be an open set of $X \times Y$ and Let $x \times y$ is a point of W.

By def of the product topology $X \times Y$.

we can find a basic element $U \times V$ such that

$$x \times y \in U \times V \subset W \longrightarrow (1)$$

where U and V are open subset of X and Y respectively.

Now $\mathcal{B}$ is a basis of X and $x \in U$ is a open in X. then there exist some basis element $B \in \mathcal{B}$ such that.

$$x \in B \subset U \longrightarrow (2)$$

111ly

$\mathcal{C}$ is a basis of Y and $y \in V$ is open in Y then there exist some basis element $C \in \mathcal{C}$ such that

$$y \in C \subset V \longrightarrow (3)$$

then (2) and (3) we get

$$x \times y \in B \times C \subset U \times V \longrightarrow (4)$$

By eqn (1) and (4) we get

$$x \times y \in B \times C \subset W$$

Thus the collection $\mathcal{J}$ meets the criterian of the lemma 13.2.

" Let X be a topological spaces. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X) and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$ then $\mathcal{C}$ is a basis for the topology of X. "

$\mathcal{D}$ is a basis of the topology of $X \times Y$.

Hence the proof.

Eg:

The product of standard topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Projection :

Let $\pi_1 : X \times Y \to X$ we defined as $\pi_1(x, y) = x$.

Let $\pi_2 : X \times Y \to Y$ we defined by the eqn

$$\pi_2(x, y) = y.$$

The maps π_1, π_2 are called projection of $X \times Y$ onto its 1st and 2nd factors respectively.

Note :

If U is an open subset of X then the set π_1^{-1}, $\pi_1^{-1}(U) = U \times Y$ which is open in $X \times Y$,

Illy

If V is an open subset of Y then the set $\pi_2^{-1}(V) = X \times V$ which is open in $X \times Y$.

U.Q (X) Thm : 15.2.

The collection $\mathcal{S} = \{\pi^{-1}(U) / U \text{ is open in } X\} \cup \{\pi^{-1}(V) / V \text{ is open in } Y\}$ is a subbasis for the product of topology on $X \times Y$.

Proof :

Let $\mathcal{T}$ denote the product topology on $X \times Y$.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

We have to know that

$$\mathcal{T} = \mathcal{T}'$$

$\because$ Every element of $\mathcal{S}$ belongs to $\mathcal{T}$.

clearly, all unions of finite intersection elements of $\mathcal{S}$ also belongs to $\mathcal{T}$.

Thus $\mathcal{T}' \supset \mathcal{T} \longrightarrow (1)$

On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of element of $\mathcal{S}$.

$\because U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, $U \times V \in \mathcal{T}'$

it follows that $U \times V \in \mathcal{T}'$

$$\Rightarrow \mathcal{T} \subset \mathcal{T}' \longrightarrow (2)$$

From eqn (1) & (2)

$$\mathcal{T} = \mathcal{T}'$$

Hence the proof.

Sec 16.

The Subspace Topology

Subspace Topology:

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$\mathcal{T}_Y = \{Y \cap U / U \in \mathcal{T}\}$ is a topology on X are the called the Subspace Topology.

With this topology, Y is called a subspace of X.

Its open set consists of all intersections of open sets of X with Y.

Lemma : 16.1.

Let $\mathcal{B}$ is a basis for the topology of X then the Collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Proof:

Let U be Open in X and

Gn: $y \in U \cap Y$

If $y \in U$ is open in X and $\mathcal{B}$ is a basis of X then $\exists$ some $B \in \mathcal{B}$, such that $y \in B \subset U$.

Then $y \in B \cap Y \subset U \cap Y$.

where $B \cap Y \in \mathcal{B}_Y$

By Lemma (13.2).

$\mathcal{B}_Y$ is a basis for a subspace topology on Y. Hence Proved.

Note:

Every open set in Y is also open in X

Lemma 16.2

Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

Proof:

Gn: U is Open in Y.

then $U = Y \cap V$ for some set V open in X.

$\because$ Y and V are both in X.

$\Rightarrow$ $Y \cap V$ is open in X

i.e. U is open in X.

Hence proved.

5M Q (F)

Thm 16.3

If A is a subspaces of X and B is a subspace of Y then the product topology on $A \times B$ is same as topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof :

Basis for the product topology on

$X \times Y = \{ U \times V / U \text{ is open in } X, V \text{ is open in } Y \}$

Basis for the subspace topology on

$A \times B = \{ (U \times V) \cap (A \times B) / U \text{ is open in } X, V \text{ is open in } Y \}$

Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$

$\because$ $U \cap A$ and $V \cap B$ are the general open set for the subspace topology on A and B respectively.

$(U \cap A) \times (V \cap B)$ is the general basis element for the product topology $A \times B$.

We conclude that the basis for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same.

Hence the topology are same.

$\therefore$ $A \times B$ inherits as a subspace of $X \times Y$.

Hence proved.

Convex :

Given an ordered set X, let us say that a subset Y on X is convex in X. If for each pair of point $a<b$ of Y, the entire integral (a,b) of points of X lie in Y.

10M (Q.Q)

Thm: 16.4

Let X be an ordered in the ordered topology : Let Y be subset of X i.e. convex in X then the ordered topology on Y is same as the ordered topology on Y inherits the subspace of X.

Proof :

Consider the ray $(a,+\infty)$ in X.

Let $a \in Y$ then

$$(a,+\infty) \cap Y = \{x / x \in Y \text{ and } x > a\}$$

This is an open ray of an ordered set Y.

If $a \notin Y$ then 'a' is either a Lower bound on Y or an upper bound on Y, since Y is convex.

The First case, the set $(a,+\infty) \cap Y$ equals all of Y.

In the Later case, its empty.

A similar remark show that the intersection of the ray $(-\infty, a)$ with Y is either an open ray of Y or Y itself.

∵ The set $(a,+\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis topology on Y. and since each is open in the ordered topology.

∴ The ordered topology contains the subspace topology.

Next we have to prove ordered topology contained in subspace topology.

W.K.T

Any open ray of Y equal the intersection of an open ray of X with Y.

So it is open in the subspace topology on Y.

∴ The open rays of Y are a subbasis for the ordered topology on Y.

This topology is contained in the subspace topology.

Hence the proof.

$U = \{1,2,3\}$
$A = \{1,2\}$
$B = \{2,3\}$
$U - A = \{3\}$ is open
A is closed

Sec 17:

Closed sets and limit points

Closed Sets

A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Eg:

1. The subset $[a,b]$ of $\mathbb{R}$ is closed.

∵ $\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$ is open.

2. In the discrete topology on the set X, every set is open it follows that every set is closed as well.

U.Q ⊗ 10M

Thm 17.1

Let X be a topological space then the following conditions holds.

i). ϕ and X are closed.

ii) arbitrary intersection of closed set are closed. (smalled positive value.)

iii). Finite union of closed sets are closed.

Proof:

i). ϕ and X are closed because there are complements of the open sets ϕ and X respectively.

ii). Let $\{A_\alpha\}_{\alpha \in J}$ be the collection of closed sets in X.

To prove: Intersection $\{A_\alpha\}$ is closed.

$\Rightarrow$ each $X - A_\alpha$ each open in X.

$\cup (X - A_\alpha)$ is open in $X \longrightarrow (1)$

By demaugon's Law.

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

$\Rightarrow X - \bigcap_{\alpha \in J} A_\alpha$ is open in X. (since by eqn (1))

$\Rightarrow \bigcap_{\alpha \in J} A_\alpha$ is closed

i.e). Arbitary intersection of closed set are closed.

(iii) Let A_i is closed for $i = 1, 2, \dots n$

T.P : $\bigcup_{i=1}^{n} A_i$ is closed in X.

$\because$ each A_i is closed.

$\Rightarrow X - A_1, X - A_2, \dots, X - A_n$ are Open.

$\Rightarrow \bigcap_{i=1}^{n} (X - A_i)$ is Open $\longrightarrow$ (2)

By demorgon's law,

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

$\Rightarrow X - \bigcup_{i=1}^{n} A_i$ is open in X ($\because$ by eqn (2))

$\Rightarrow \bigcup_{i=1}^{n} A_i$ is closed.

i.e., Finite unions of closed sets are closed.

Hence proved.

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Thm 17:2.

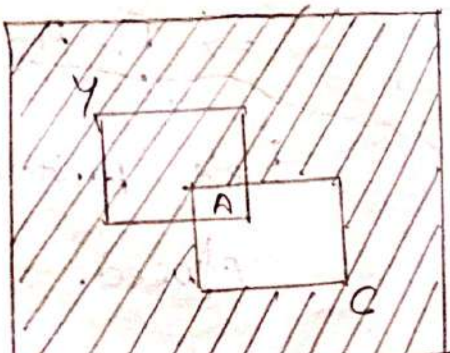
Let Y be a subspace of X. then a set A is closed in Y iff it equals to the intersection of a closed set with Y.

Proof:

Let us assume that

$A = C \cap Y$ where C is closed in X

T.P : A is closed in Y



$\because$ C is closed in X.

$\Rightarrow X - C$ is open in X

$\Rightarrow X - C \cap Y$ is open in Y

By the def of the subspace topology.

But $X - C \cap Y = (X \cap Y) - (C \cap Y)$

$= Y - A$

Hence $Y - A$ is open in Y.

$\Rightarrow A$ is closed in Y.

Conversely,

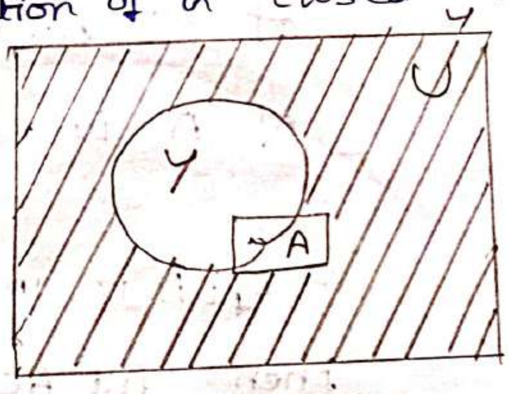
Assume that A is closed in Y then $Y - A$ is open in Y.

By def, $Y - A = U \cap Y$, for some set U of X.

The set for some $X - U$ is closed in X and

$A = Y \cap (X - U)$.

So that A is the equals intersection of a closed set of X with Y.



Thm 17.3.

Let Y be a subspaces of X if A is closed in Y and Y is closed in X then A is closed in X.

Proof:

A is closed in Y.

$\Rightarrow A = C \cap Y$

Where C is closed in X

Gn: Y is closed in X

By thm 17.2.

The intersection C in Y is also closed in X.

$\therefore$ A is closed in X

Hence the proof.

Interior of a set:

If A is a subset of a topological space X, the interior of A is defined as the union of all open sets contained in A. The interior of A is denoted by Int A.

Closure of a set:

If A is a subset of a topological space X, the closure of A is defined as the intersection of all closed sets containing in A. The closure of A is denoted by Cl. A (or) $\bar{A}$.

Note:

i) Int A is an open set and $\bar{A}$ is a closed and then int $A \subset A \subset \bar{A}$.

ii) If A is open, A = int A, why If A is closed, $A = \bar{A}$.

Thm 17.4 X

Let Y be a subspace of X; Let A be a subset of Y; Let $\bar{A}$ be denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof:

Let B denote the closure of A in Y.

T.P: $B = \bar{A} \cap Y$

Now given the set $\bar{A}$ is closed in X.

$\Rightarrow \bar{A} \cap Y$ is closed in Y. (by 17.2)

For $A \subset \bar{A}$, $A \subset Y \Rightarrow A \subset \bar{A} \cap Y$.

$\because$ By definition, B equals the intersection of all closed subsets of Y containing A.

$\therefore B \subset \bar{A} \cap Y \longrightarrow (1)$

Conversely

W.K.T,

B is closed in Y.

Hence by thm (17.2)

$B = C \cap Y$ for some set C closed in X.

Then C is closed set of X containing A.

$\because \bar{A}$ is the intersection of all such closed set.

$\therefore \bar{A} \subset C$

$\Rightarrow (\bar{A} \cap Y) \subset (C \cap Y)$

$\Rightarrow \bar{A} \cap Y \subset B \longrightarrow (2)$

From (1) & (2)

$B = \bar{A} \cap Y$.

Hence the closure of A equals $\bar{A} \cap Y$.

Hence proved.

U.Q 5M 10M Thm 17.5:

Let A be a subset of the topological space X. Then a). $x \in \bar{A}$ iff every open set U containing x intersect A.

b). Supposing the topology of X is given by basis then $x \in \bar{A}$ iff every basis element B containing x intersect A.

Proof :

A). We shall prove that, by transform each implécation to its contra positive.

To prove : $x \notin \bar{A}$ iff ~~$x \notin \bar{A}$~~ $\exists$ an open set U containing x that doesn't intersect A.

Suppose $x \notin \bar{A}$ then $x \in X - \bar{A}$.

$\Rightarrow X - \bar{A}$ is open

~~Convers.~~

Thus, there exist an open set $U = X - \bar{A}$ is an open set containing x that doesn't intersect A.

Conversely,

If $\exists$ an open set U containing x which doesn't intersect A then $X - U$ is a closed set containing A.

By defn,

$\bar{A}$ is intersection of all closed sets containing A.

It follows that $\bar{A} \subset X - U$

$\therefore x \in U$, $x \notin \bar{A}$

This proves 'A' part.

B). If every set containing x intersection A.

Let $\mathcal{B}$ be a basis of X, then there exist some $B \in \mathcal{B}$ be a basis of such that $x \in B \subset U$

by part 'A',

It follows that $x \in \bar{A}$ iff every basis element B containing x intersect A.

∴ This proof 'B' part.

Hence proved.

Def : Neighbourhood :

If U is an open set containing x, it can be ~~return~~ written as U is a neighbourhood x

Note:

We can reach state the 1st part of the above theorem as follows.

If A is a subset of the topological space X, then $x \in \bar{A}$ iff every neighbourhood of X, intersect A.

U.Q 2M

Def: Limit points:

If A is a subset of the topological space X, and if x is a point of X then x is a Limit point (or) cluster point (or) point of accumulation of A if every neighbourhood X intersect A in some points other than x itself.

SM U.Q
Q7
2-

Thm 17.6.

Let A be a subset of a topological space X. Let A' be the set of all limit points of A then $\overline{A} = A \cup A'$

Proof

Suppose if x is in A', every neighbourhood of x intersect A.

By thm 17.5

$x \in \overline{A}$, hence $A' \subset \overline{A}$

$\because$ by def

$A \cup A' \subset \overline{A} \longrightarrow ①$

Conversely,

We now prove $\overline{A} \subset A \cup A'$

Let $x \in \overline{A}$ and show that $x \in A \cup A'$

suppose that x doesn't lie in A.

$\therefore x \in \overline{A}$

$\Rightarrow$ neighbourhood of U of x intersect A.

$\because x \notin \overline{A}$, $U \cap A$ must contained a point

$\Rightarrow x$ is a limit point of A

$\Rightarrow x \in A'$ and $x \in A \cup A'$ thus $\overline{A} \subset A \cup A' \rightarrow ②$

From ① & ②

$\overline{A} = A \cup A'$

Corollary 17.7.

A subset of a topological space is closed iff it contains all its limit points.

Proof :

By def of closure ..

W.K.T

A is closed iff $A = \bar{A}$

By above thm, $\bar{A} = A \cup A'$

A is closed iff $A = A \cup A'$

$\Longleftrightarrow A' \subset A$

$\Longleftrightarrow$ Every limit point of A' is in A.

Hence the proof.

U.Q 2m Def: Hausdroff space:

A Topological space X is called a Hausdroff space if for each pair (x_1, x_2) of distinct points of X their exist neighbourhood U_1 and U_2 of x_1, x_2 respectively that are disjoint.

U.Q 5M Thm 17.8

Every finite point set in a hausdroff space is closed.

Proof :

It is enough to prove that every 1-point set $\{x_0\}$ is closed.

Consider the 1-point set $\{x_0\}$.

If x is a point of X different from x_0, then x, x_0 have disjoint neighbourhoods U and V respectively,

$\because x \in U,\ x \neq x_0$

$\Rightarrow U$ interesect $\{x_0\} = \phi$ ($x \cap \{x_0\}$)

$\Rightarrow x \notin \overline{\{x_0\}}$

$\{x_0\} = \overline{\{x_0\}}$ [$\because$ by thm 17.5]

$\{x_0\}$ is closed.

Hence proved.

Note:

The condition that finite point sets are closed are called as T_1 axiom.

Thm 17.9.

Let X be a space satisfying the T_1 axiom, Let A be a subset of X then the points x is a Limit point of A iff every neighbourhood of X contains if infinitely many points of A.

Proof:

If every neighbourhood of x intersect A in infinitely many points then it must intersect A in some points other than x itself.

$\Rightarrow x$ is a limit point of A,

Some points other than x itself.

Conversely,

Suppose x is a limit point of A.

T.P: Every neighbourhood of x contains infinitely many points of A.

Let us suppose some neighbourhood U of x intersect A in only finitely many points.

i.e. $U \cap A$ is finitely many points.

Then U is also intersect $A - \{x\}$ in finitely many points.

Let $\{x_1, x_2 \dots x_m\}$ be the points of $U \cap (A - \{x\})$

$\because X$ is hausdroff then the set $\{x_1, x_2 \dots x_m\}$ is closed.

$\therefore$ Its Complement is open.

$\Rightarrow X - \{x_1, x_2 \dots x_m\}$

This an open set of X.

Then

$V = U \cap \{X - \{x_1, x_2 \dots x_m\})$ is a neighbour-hood of x and $V \cap (A - \{x\}) = \phi$.

This contradicts the assumption, x is a limit point of A.

Thus if x is a limit point of A then every neighbourhood of x contains infintely many points of A.

Hence the proof.

Thm 17.10

If X is a hausdroff space, then a sequence of points X converges to atmost one point of X.

Proof:

Let x_n is a sequence of points of X.

T.P: The sequence x_n is converges to atmost one point of X.

Suppose that x_n is a sequence of points of X that converges to x.

If $y \neq x$ and let U be disjoint neighbourhood x and y respectively.

∵ U contains x_n for all

But finitely many values of n and the set V cann't contains any ~~set~~ sequence of point of x_n

∴ x_n cann't converges y.

Hence a sequence of points of X converges to atmost one point of X.

Hence the proof.

Exercise

Thm 17.11

i) Every simply ordered set is a hausdroff space in the ordered topology.

ii) The product of two hausdroff space is a hausdroff space.

iii) A subspace of a hausdroff space is a hausdroff space.

Proof:

i). Let X is a simply ordered set under the relation $<$.

T.P: X is a hausdroff in the order topology.

Let $x \neq y \in X$ where $x < y$

If $z \in X$ such that $x < z < y$

If $a_0, b_0 \in X$ where a_0, b_0 are the smallest and largest element of X.

Take $U = [a_0, y)$, $V = (x, b_0]$

Case (i)

If $a_0 \in X$ but $b_0 \notin X$

Take $U = [a_0, y)$, $V = (x, \infty)$

Case (ii)

If $a_0 \notin X$, $b_0 \in X$

Take $U = (-\infty, y)$ & $V = (x, b_0]$.

Case (iii)

If $a_0 \notin X$, $b_0 \notin X$

Take $U = (-\infty, y)$ & $V = (x, \infty]$.

$\therefore$ Every simply order set in a hausdorff space

ii) Let $(X_\alpha)_{\alpha \in I}$ be a hausdroff space.

T.P: $\prod_{\alpha \in I} X_\alpha$ is a hausdraff space.

Let (x, y) be two distinct point in $\prod X_\alpha$ then

$x = (x_\alpha)_{\alpha \in I}$ and $y = (y_\alpha)_{\alpha \in I}$, $x \neq y$.

$\Rightarrow x_\beta \neq y_\beta$ for some $\beta \in I$

i.e) x_β, y_β are two distinct points in A which is hausdraff.

Then $\exists$ U_β, V_β are neighbourhood of x_β and y_β respectively. Such that

$$U_\beta \cap V_\beta = \phi \longrightarrow (1)$$

Let $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are subbasic set for the product and so are open πX_α for $\alpha \in I$.

$\pi_\beta(x) = x_\beta \in U_\beta$

$\Rightarrow x \in \pi_\beta^{-1}(x_\beta)$

$\Rightarrow x \in \pi_\beta^{-1}(U_\beta)$

III^{ly}

$y \in \pi_\beta^{-1}(V_\beta)$

Now (1) $\Rightarrow$

$\pi_\beta^{-1}(U_\beta \cap V_\beta) = \pi_\beta^{-1}(\phi)$

$1.1 = 1$
$0.0 = 0$
$\phi\,\phi = \phi$

$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \phi$

Thus $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are neighbourhood of x and y respectively and are distinct.

$\therefore \pi X_\alpha$ is hausdroff space.

iii). Let X be a Hausdroff Space.

Let Y be a Subspace of X.

T.P: Y is a Hausdroff space.

Let $y_1 \neq y_2$ be any two distinct point in Y then y_1, y_2 are two distinct points in X which is hausdroff,

$\therefore$ If U_1, U_2 are open in X such that $y_1 \in U_1$ and $y_2 \in U_2$ and $U_1 \cap U_2 = \phi$.

$\because$ $U_1 \cap Y$ and $U_2 \cap Y$ are open in Y

$\Rightarrow$ $y_1 \in U_1 \cap Y$ and $y_2 \in U_2 \cap Y$ such that

$$(U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y$$
$$= \phi \cap Y.$$

$$(U_1 \cap Y) \cap (U_2 \cap Y) = \phi$$

Hence Y is a Hausdroff space.

Thm : 17.~~12~~

Show that the collection of open rays in an ordered set on A is a subbasis for the ordered topology on A.

Proof :

T.P : The open rays from a subbasis for the ordered topology on A.

$\because$ The open rays are open in the ordered topology. The topology they generate is contain in the ordered topology on the other hand every basis element for the ordered topology equals a finite intersection of open rays

The intervals $(a, b) = (-\infty, b) \cap (a, +\infty)$ while $[a_0, b]$ and $(a, b_0]$, if there are themselves open rays.

Hence the topology generates by the open rays contains the ordered topology.

Hence the collection of open rays in an ordered set A is a subbasis for the ordered topology on A.

Hence the proof.

Thm : 17.15

P.T Subspace topology is a topology.

Proof:

T.P : The subspace topology

$J_y = \{ Y \cap U / U \in J \}$ is a topology.

(i) since $\phi = Y \cap \phi$ and $Y \cap X$.

$\therefore$ ϕ and Y are topology in J_y.

(ii) By def $Y \cap U_\alpha \in J_y$ $\forall \alpha \in J$

$\therefore$ $\cup_\alpha (Y \cap U_\alpha) = Y \cap (\cup_\alpha U_\alpha) \in J_y$

$\because$ U_α is open in X for all α.

$\Rightarrow$ $\cup U_\alpha$ is open in X.

(iii) $Y \cap U_1, Y \cap U_2 \dots Y \cap U_n \in J_y$.

$\Rightarrow (Y \cap U_1) \cap (Y \cap U_2) \cap \dots \cap (Y \cap U_n)$

$\Rightarrow Y \cap (U_1 \cap U_2 \cap \dots \cap U_n)$

$\Rightarrow Y \cap (\bigcap_{i=1}^{n} U_i) \in J_y$

$\therefore$ J_y is a topology on Y.

Additional Resource :

http://mathforum.org

http://ocw.mit.edu/ocwweb/Mathematics

http://www.opensource.org

http://en.wikipedia.org

Practice Questions:

Question Bank

Section – A

1.Define Topology.
2.Define finer.
3.Define Basis.
4.Define Standard Topology.
5.Define k Topology.
6.Define Order Topology.
7.Dfine Rays.
8.Define Product Toplogy.
9. What is Projections?
10.Define Subspace topology.
11.Define limit points.
12Define Hausdorff Space.

Section – B

1. 1.If $\mathfrak{B}\ and\ \mathfrak{B}'$ are basis for the topologies $\tau\ and\ \tau'$ respectively on a set ,then show that the following are equivalent

   (i) $\tau'\ is\ finer\ than\ \tau$

   (ii) For each $x \in X$ and each bais elt $B \in \mathscr{B}$ containing $x, there\ is\ a\ B' \in \mathfrak{B}'$ such that $x \epsilon B' \subset \mathfrak{B}$
2. Define subspace topology for a subset $Y\ of\ X$ .If Y is aubspace of X then S.T a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y
3. If $\mathfrak{B}$ is a basis for the topology of X,P.T the collection $\mathfrak{B}_Y = \{B \cap Y\ / \in \mathfrak{B}\} is\ a\ basis$ for the subspace topology on Y
4. Prove that every simply ordered set is a Hausdroff space in the order topology
5. Let X be a set,let $\mathfrak{B}$ be a basis for a topology τ on X.P.T τ equals the collection of all union of elements of c
6. Let A be a subset of the topological space X,let A' be the set of all limit points of A.P.T $\bar{A} = A \cup A'$
7. Define standard topology and lower limit topology.P.T the lower limit topology is finer than the standard topology
8. Define open rays in the ordered set A.Show that the collection of open rays in an ordered set $A$is a subbasis for the ordered topology on A

9. Show that the collection $s = \{\pi_1^{-1}\ (U)/U\ is\ open\ in\ X\} \cup \quad \pi_2^{-1}\ V/V\ is\ \ open\ Y\}$

$$is\ \ a\ subbasis\ for\ the\ \ product\ topology\ on\ X \times Y$$

10. Define a basis for a topology on a set X.
11. Define product topology on $X \times Y$, where X & Y are topological spaces
12. If $\mathfrak{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology of Y.P.T the collection $D = \{B \times C/B \in\ \mathfrak{B}\ and$ C$\in \mathcal{C}\}$ is a basis for the topology of $X \times Y$
13. If A is a subspace of X and B is a subspace of Y, Prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$
14. Let Y be a subspace of X; let A be a subset of Y ;let $\bar{A}$ denote the closure of A in X,prove that the closure of A in Y equals $\bar{A}\ \cap Y$
15. Define Haussdorff space, Prove that every finite point set in a Haussdorff space is closed.
16. Define Subspace topology, verify that it is a topology

Section – C

1 .Let A be a subset of the topological space X, then prove that$x \in \bar{\bar{A}}$ if and only if every open set U Containing x intersects A Supposing the topology of X is given by a basis,then $x \in \bar{\bar{A}}$ if and only basis element B Containing x intersects A

2.a) Let Y be a subspace of X. Prove that a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y

b) let A be a subset of Y ;let $\bar{A}$ denote the closure of A in X,prove that the closure of A in Y equals $\bar{\bar{A}}\ \cap Y$

3.Let X be a topological space, P.T the following conditions hold

a) $\emptyset$ and X are closed

b) Arbitrary intersection of closed sets are closed

c) Finite union of closed sets are closed.

Recommended Text : James R. Munkres, Topology (2nd Edition) Pearson Education Pve. Ltd., Delhi-2002 (Third Indian Reprint)